

# GENERALIZED RANDOM ENERGY MODEL AT COMPLEX TEMPERATURES

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**ABSTRACT.** Motivated by the Lee–Yang approach to phase transitions, we study the partition function of the Generalized Random Energy Model (GREM) at *complex* inverse temperature  $\beta$ . We compute the limiting log-partition function and describe the fluctuations of the partition function. For the GREM with  $d$  levels, in total, there are  $\frac{1}{2}(d+1)(d+2)$  phases, each of which can symbolically be encoded as  $G^{d_1}F^{d_2}E^{d_3}$  with  $d_1, d_2, d_3 \in \mathbb{N}_0$  such that  $d_1 + d_2 + d_3 = d$ . In phase  $G^{d_1}F^{d_2}E^{d_3}$ , the first  $d_1$  levels (counting from the root of the GREM tree) are in the *glassy* phase (G), the next  $d_2$  levels are dominated by *fluctuations* (F), and the last  $d_3$  levels are dominated by the *expectation* (E). Only the phases of the form  $G^{d_1}E^{d_3}$  intersect the real  $\beta$  axis. We describe the limiting distribution of the zeros of the partition function in the complex  $\beta$  plane (= Fisher zeros). It turns out that the complex zeros densely touch the positive real axis at  $d$  points at which the GREM is known to undergo phase transitions. Our results confirm rigorously and considerably extend the replica-method predictions from the physics literature.

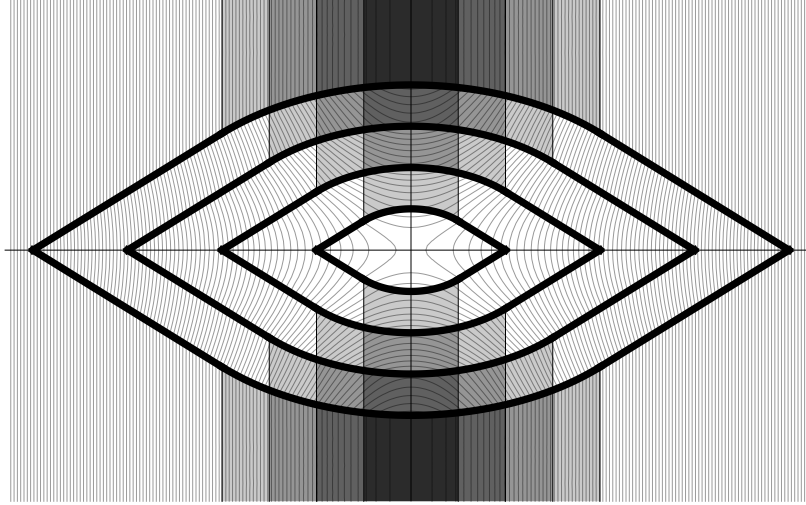


FIGURE 1. Phase diagram of the GREM in the complex  $\beta$  plane together with the level lines of the limiting log-partition function. See Figure 4 for details.

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*Key words and phrases.* Generalized Random Energy Model, complex inverse temperature, spin glasses, partition function, Lee–Yang program, Fisher zeros, extreme values, Poisson cascade zeta function, lines of zeros.

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## 1. INTRODUCTION AND DEFINITION OF THE MODEL

**1.1. Introduction.** The study of phase transitions is one of the central topics in statistical physics. Phase transitions are usually defined as the values of physical parameters (for example, the inverse temperature  $\beta$ ) at which the *limiting log-partition function* (or equivalently the *free energy*) is not real analytic (= non-analytic in any neighborhood of the phase transition point). However, at any *finite* system size, the log-partition function is real analytic. In order to explain why the infinite system log-partition function loses analyticity (while the finite system log-partition function does not), Lee and Yang [48, 29] suggested to look at *complex* values of the inverse temperature  $\beta$ . At complex temperatures, the partition function may have *zeros* and hence, the log-partition function has singularities, even for finite system sizes. If in the infinite system limit these singularities accumulate around the real axis at some  $\beta_c \in \mathbb{R}$ , then the limiting log-partition function may lose analyticity at  $\beta_c$ , even though  $\beta_c$  itself is never a point of singularity of the log-partition function. Thus, the approach of Lee and Yang relates phase transitions to the distribution of *complex zeros* of the partition function. The study of the complex zeros of the partition function is usually referred to as the Lee–Yang program; see for example [3, 4], where a large class of lattice spin models is considered from this point of view.

The aim of the present work is to study a special model of spin glass, the *Generalized Random Energy Model* (GREM) within the Lee–Yang program. The simplest model of a spin glass is the *Random Energy Model* (REM) introduced by Derrida [12, 13]. In this model, the energies of the system are assumed to be *independent* Gaussian random variables. The behavior of the REM at real inverse temperature is well understood; see Bovier et al. [10] and Bovier [5, Chapter 9]. For the REM at *complex* inverse temperature, Derrida [15] derived the limiting free energy, obtained the phase diagram and computed the limiting distribution of complex zeros of the partition function. The present authors refined Derrida’s results and provided rigorous proofs in [25].

Although the REM contains some of the physics of the spin glasses, e.g., it displays the freezing phenomenon, the REM does not exhibit such phenomena as multiple freezing transitions and chaos which are observed, e.g., in the celebrated Sherrington–Kirkpatrick (SK) model of a spin glass. In order to obtain a solvable model with multiple freezing transitions, Derrida introduced the *Generalized Random Energy Model* (GREM); see [14, 16, 17]. Rigorous results on the GREM at real inverse temperatures were obtained by Capocaccia et al. [11] and in a series of works by Bovier and Kurkova [7, 8, 6]. For a review of these results, we refer to Bovier and Kurkova [9] and Bovier [5, Chapter 10]. We note in passing that the recent progress in rigorous understanding of the SK model draws heavily on the analysis of the GREM, see [34] for a review.

In the theoretical physics literature, there is a strong interest in studying spin glass models at complex temperatures. Besides the Lee–Yang program, the motivation comes here from quantum physics and concretely from the studies of interference in inhomogeneous media. See, e.g., the recent works of Takahashi and Obuchi [33, 45, 46], Saakian [40, 41], Dobrinevski et al. [19]. In particular, Takahashi [45], developed a complex version of the (non-rigorous) replica method and used it to identify the phase diagram of the GREM.

As for the rigorous works, beyond the uncorrelated case of the REM, to our knowledge, only two models of disordered systems with correlated complex random energies have been studied to some extent: the Branching Random Walk and the Gaussian Multiplicative Chaos. See Derrida et al. [18] and the recent works of Lacoïn et al. [27] and Madaule et al. [30, 31]. Both models have correlations of logarithmic type and their complex-plane phase diagrams are quite similar to that of the REM (see Section 2.10 for more details).

The GREM seems to be a natural candidate to be tackled next from the Lee–Yang viewpoint. On the one hand, as we show below, the complex GREM is a rather tractable model even at the level of fluctuations, and, on the other hand, it exhibits multiple freezing phase transitions and has a much richer phase diagram than that of the REM.

The main results of this paper can be summarized as follows:

- (1) we compute the limiting log-partition function  $p(\beta) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)|$ ;
- (2) we describe the global limiting distribution of complex zeros of  $\mathcal{Z}_n(\beta)$ ;
- (3) we identify the limiting fluctuations of  $\mathcal{Z}_n(\beta)$ ;
- (4) we prove functional limit theorems for  $\mathcal{Z}_n(\beta)$  in a suitably rescaled neighborhood of a fixed  $\beta_* \in \mathbb{C}$ ;
- (5) we describe the local limiting distribution of complex zeros of  $\mathcal{Z}_n(\beta)$  in a suitably rescaled neighborhood of a fixed  $\beta_* \in \mathbb{C}$ .

These results give the complete phase diagram of the GREM; see Figures 1 and 4. Our results confirm the replica-method predictions of Takahashi [45] and extend these considerably. We also indicate how to pass to the limit of continuous hierarchies (*Continuous Random Energy Model*, CREM), see Section 2.10, which allows us to compare our results with the ones on the Branching Random Walk [30] and the Gaussian Multiplicative Chaos [27, 31]. We hope that our results shed more light on the complex plane phase diagrams and on fluctuations in strongly correlated random energy models.

**1.2. Notation: Definition of the GREM.** We start by introducing the notation which will be used throughout the paper. Fix the following parameters:

- (1) the *number of levels*  $d \in \mathbb{N}$ ;
- (2) the *variances* of the levels  $a_1, \dots, a_d > 0$  (energetic parameters);
- (3) the *branching exponents*  $\alpha_1, \dots, \alpha_d > 1$  (entropic parameters).

We also fix  $d$  sequences  $\{N_{n,1}\}_{n \in \mathbb{N}}, \dots, \{N_{n,d}\}_{n \in \mathbb{N}}$  of natural numbers (called the *branching numbers*) such that for every  $1 \leq j \leq d$ ,

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{N_{n,j}}{\alpha_j^n} = 1.$$

The reader may simply take  $N_{n,1} = [\alpha_1^n], \dots, N_{n,d} = [\alpha_d^n]$ . Consider a *rooted tree*, denoted by  $\mathbb{T}_n$ , which is constructed in the following way. The root of the tree is located at level 1 and is connected by edges to  $N_{n,1}$  vertices (descendants) at level 2. Any vertex at level 2 is connected to  $N_{n,2}$  vertices at level 3, and so on. Finally, any vertex at level  $d$  is connected to  $N_{n,d}$  terminal vertices (leaf nodes) which have no descendants. We label the edges of the tree by  $d$  levels  $1, \dots, d$  so that the edges issuing from the root are at level 1, whereas the leaf edges of the tree are at level  $d$ . The set of paths in  $\mathbb{T}_n$  connecting the root to the terminal vertices is denoted by

$$(1.2) \quad \mathbb{S}_n = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{N}^d : 1 \leq \varepsilon_1 \leq N_{n,1}, \dots, 1 \leq \varepsilon_d \leq N_{n,d}\}.$$

The total number of elements in  $\mathbb{S}_n$  and its growth exponent are given by

$$(1.3) \quad N_n := \#\mathbb{S}_n = N_{n,1} \cdot N_{n,2} \cdot \dots \cdot N_{n,d} \sim \alpha^n, \quad \alpha := \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_d.$$

Consider independent real standard normal random variables attached to the edges of the tree and denoted by

$$(1.4) \quad \{\xi_{\varepsilon_1 \dots \varepsilon_j} : 1 \leq j \leq d, 1 \leq \varepsilon_1 \leq N_{n,1}, \dots, 1 \leq \varepsilon_j \leq N_{n,j}\}.$$

Define a zero-mean Gaussian random field  $X = \{X_\varepsilon : \varepsilon \in \mathbb{S}_n\}$  by

$$(1.5) \quad X_\varepsilon = \sqrt{a_1} \xi_{\varepsilon_1} + \sqrt{a_2} \xi_{\varepsilon_1 \varepsilon_2} + \dots + \sqrt{a_d} \xi_{\varepsilon_1 \dots \varepsilon_d}.$$

Note that the variance of this random field is constant:

$$(1.6) \quad a := a_1 + \dots + a_d = \text{Var } X_\varepsilon, \quad \varepsilon \in \mathbb{S}_n.$$

In the literature on the GREM, one usually assumes that the total number of energies in  $\mathbb{S}_n$  is  $N_n = 2^n$  (so that  $\alpha = \log 2$ ) and that the variance is  $a = 1$ . Since we will often use induction over the number of levels of the GREM, it is more convenient to us to consider the general case omitting these assumptions.

Let us write the complex inverse temperature  $\beta$  in the form

$$\beta = \sigma + i\tau \in \mathbb{C}, \quad \sigma = \text{Re } \beta \in \mathbb{R}, \quad \tau = \text{Im } \beta \in \mathbb{R}.$$

The *partition function* of the Generalized Random Energy Model at inverse temperature  $\beta \in \mathbb{C}$  is defined by

$$(1.7) \quad \mathcal{Z}_n(\beta) = \sum_{\varepsilon \in \mathbb{S}_n} e^{\beta \sqrt{n} X_\varepsilon}.$$

Define the *critical inverse temperatures*

$$(1.8) \quad \sigma_j = \sqrt{\frac{2 \log \alpha_j}{a_j}} \in \mathbb{R}, \quad 1 \leq j \leq d.$$

To make the notation consistent, we make the convention  $\sigma_0 = 0$  and  $\sigma_{d+1} = +\infty$ . Throughout the whole paper, we assume that

$$(1.9) \quad \sigma_1 < \dots < \sigma_d.$$

Geometrically, this condition means that the broken line joining the points

$$(a_1 + \dots + a_j, \log \alpha_1 + \dots + \log \alpha_j), \quad 0 \leq j \leq d,$$

is strictly concave. If (1.9) is not satisfied, one has to coarse grain the GREM levels by replacing the above broken line by its concave hull; see [7] for details in the real  $\beta$  case. If (1.9) does not hold, there are less phase transition temperatures than  $d$ . In order to avoid complicated notation, we assume (1.9).

Often, we can restrict ourselves to the quarter-plane  $\sigma \geq 0$  and  $\tau \geq 0$  because of the straightforward distributional equalities

$$(1.10) \quad \{\mathcal{Z}_n(-\beta) : \beta \in \mathbb{C}\} \stackrel{d}{=} \{\mathcal{Z}_n(\beta) : \beta \in \mathbb{C}\},$$

$$(1.11) \quad \{\mathcal{Z}_n(\bar{\beta}) : \beta \in \mathbb{C}\} \stackrel{d}{=} \{\overline{\mathcal{Z}_n(\beta)} : \beta \in \mathbb{C}\}.$$

**1.3. Notation: Spaces and modes of convergence.** In this section, we briefly recall several notions of convergence which will be frequently used below. For more information, we refer to the classical books [2] and [26]. The reader may skip this section and return to it when necessary.

Let  $(D, \rho_D)$  be a locally compact metric space with metric  $\rho_D$ . If not stated otherwise, all measures on  $D$  are defined on the Borel  $\sigma$ -algebra generated by the metric  $\rho_D$ .

*Space of Radon measures.* A Radon measure on  $D$  is a measure  $\mu$  on  $D$  having the property that  $\mu(K) < \infty$  for every compact set  $K \subset D$ . Let  $\mathcal{M}(D)$  be the set of all Radon measures on  $D$ . A sequence of Radon measures  $\mu_1, \mu_2, \dots \in \mathcal{M}(D)$  converges *vaguely* to a Radon measure  $\mu \in \mathcal{M}(D)$  if for every continuous compactly supported function  $f: D \rightarrow \mathbb{R}$  we have  $\lim_{k \rightarrow \infty} \int_D f d\mu_k = \int_D f d\mu$ . Endowed with the topology of vague convergence,  $\mathcal{M}(D)$  becomes a Polish space. A *random measure* on  $D$  is a random variable defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and taking values in  $\mathcal{M}(D)$ .

*Space of integer-valued Radon measures.* Let  $\mathcal{N}(D)$  be the subset of  $\mathcal{M}(D)$  consisting of all measures  $\mu$  such that  $\mu(K) \in \mathbb{N}_0$  for every compact set  $K \subset D$ . Measures with this property are called *integer-valued*. Every measure  $\mu \in \mathcal{N}(D)$  can be represented as  $\mu = \sum_{i \in I} \delta(x_i)$ , where  $\{x_i\}_{i \in I}$  is at most countable collection of points in  $D$  having no accumulation points in  $D$ . Here,  $\delta(x)$  is the Dirac delta-measure at  $x \in D$ . It is well-known that  $\mathcal{N}(D)$  is a closed subset of  $\mathcal{M}(D)$ . We endow  $\mathcal{N}(D)$  with the induced vague topology. A *point process* on  $D$  is a random variable defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and taking values in  $\mathcal{N}(D)$ .

*Space of continuous functions.* Recall that  $(D, \rho_D)$  is a locally compact metric space with metric  $\rho_D$ . Let  $C(D)$  be the space of all (not necessarily bounded) continuous complex-valued functions on  $D$ . A sequence of continuous functions on  $D$  converges *locally uniformly* if it converges uniformly on every compact set  $K \subset D$ . Endowed with the topology of locally uniform convergence, the space  $C(D)$  becomes a Polish space. A *random continuous function* on  $D$  is a random variable defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and taking values in  $C(D)$ .

If  $D$  is an open subset of  $\mathbb{C}^d$ , let  $\mathcal{H}(D)$  be the set of all complex-valued functions which are analytic on  $D$ . Note that  $\mathcal{H}(D)$  is a closed linear subspace of  $C(D)$ . We endow  $\mathcal{H}(D)$  with the topology of locally uniform convergence induced from  $C(D)$ . A *random analytic function* on  $D$  is a random variable defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and taking values in  $\mathcal{H}(D)$ .

*Weak convergence.* Let  $(M, \rho_M)$  be a metric space. A sequence of random elements  $Z_1, Z_2, \dots$  taking values in  $M$  converges *weakly* to a random element  $Z$  with values in  $M$  if for every continuous, bounded function  $f: M \rightarrow \mathbb{R}$ , we have  $\lim_{k \rightarrow \infty} \mathbb{E}f(Z_k) = \mathbb{E}f(Z)$ . In the case when  $M$  is  $\mathcal{M}(D)$ ,  $\mathcal{N}(D)$ ,  $C(D)$ , or  $\mathcal{H}(D)$ , we speak of weak convergence of random measures, point processes, random continuous functions, or random analytic functions, respectively.

*Zeros of analytic functions.* For an analytic function  $f$  which is defined on some domain (=connected open set)  $D \subset \mathbb{C}$  and does not vanish identically, we denote by  $\mathbf{Zeros}\{f(\beta): \beta \in D\} \in \mathcal{N}(D)$  an integer-valued Radon measure on  $D$  which counts the zeros of  $f$  in  $D$  according to their multiplicities.

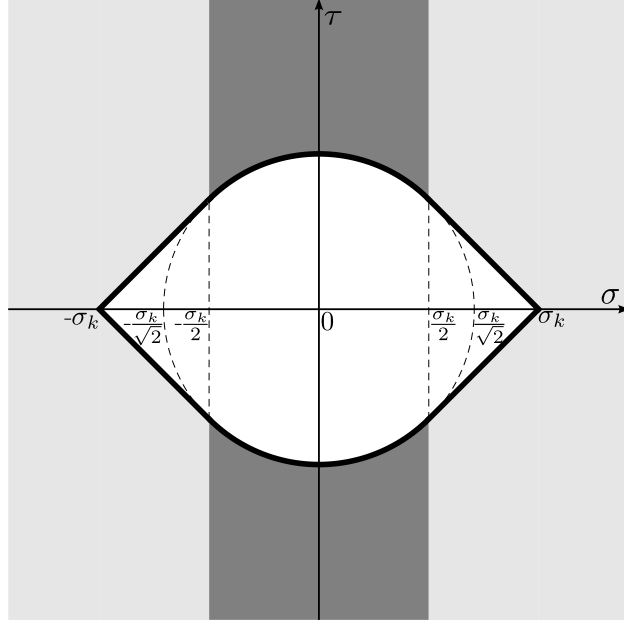


FIGURE 2. Complex  $\beta$  phase diagram of the REM with the partition function  $\mathcal{Z}_n^{(k)}(\beta)$ , see Derrida [15] and also [25].

*Real and complex Gaussian distribution.* The *real Gaussian distribution*  $N_{\mathbb{R}}(0, \theta^2)$  with mean zero and variance  $\theta^2 > 0$  has density

$$\varphi_{\mathbb{R}}(t) = \frac{1}{\sqrt{2\pi}\theta} e^{-\frac{t^2}{2\theta^2}}, \quad t \in \mathbb{R},$$

w.r.t. the Lebesgue measure on  $\mathbb{R}$ . The *complex Gaussian distribution*  $N_{\mathbb{C}}(0, \theta^2)$  with mean zero and variance  $\theta^2 > 0$  has density

$$\varphi_{\mathbb{C}}(t) = \frac{1}{\pi\theta^2} e^{-\frac{|t|^2}{\theta^2}}, \quad t \in \mathbb{C},$$

w.r.t. the Lebesgue measure on  $\mathbb{C}$ . Note that  $Z \sim N_{\mathbb{C}}(0, \theta^2)$  iff  $Z = X + iY$ , where  $X, Y \sim N_{\mathbb{R}}(0, \frac{1}{2}\theta^2)$  are independent. A zero mean real or complex Gaussian distribution is called *standard* if  $\theta = 1$ .

Throughout the paper,  $C, C_1, \dots$  denote positive constants whose values may change from line to line. Let  $\mathbb{R}_+ = (0, \infty)$ . We write  $a_n \sim b_n$  if  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ .

## 2. STATEMENT OF RESULTS

**2.1. Limiting log-partition function.** In this section, we state a formula for the limiting log-partition function of the GREM. To understand this formula heuristically, imagine a GREM with  $d$  levels as a “superposition” of  $d$  independent copies of the REM. (Note that the random field  $X_\varepsilon$  which generates the partition function of the GREM, cf. (1.7), has strong correlations.) Namely, with every level  $k = 1, \dots, d$



of the GREM we can associate a REM whose partition function is given by

$$(2.1) \quad \mathcal{Z}_n^{(k)}(\beta) = \sum_{j=1}^{N_{n,k}} e^{\beta \sqrt{n a_k} \eta_j^{(k)}}, \quad 1 \leq k \leq d,$$

where  $\eta_1^{(k)}, \eta_2^{(k)}, \dots, \eta_{N_{n,k}}^{(k)}$  are independent real standard normal random variables. The complex plane phase diagram of the REM has been described by Derrida [15]; see also [25]. There are three phases, see Figure 2, which we will denote by

- (a)  $E_k$  (*expectation* dominated phase),
- (b)  $F_k$  (*fluctuations* dominated phase),
- (c)  $G_k$  (“*glassy phase*” = extreme values dominated phase).

Concretely, the phases are given by

$$(2.2) \quad G_k = \{\beta \in \mathbb{C} : 2|\sigma| > \sigma_k, |\sigma| + |\tau| > \sigma_k\},$$

$$(2.3) \quad F_k = \{\beta \in \mathbb{C} : 2|\sigma| < \sigma_k, 2(\sigma^2 + \tau^2) > \sigma_k^2\},$$

$$(2.4) \quad E_k = \mathbb{C} \setminus \overline{G_k \cup F_k},$$

where  $\bar{A}$  is the closure of the set  $A$ . The phases  $G_k$  and  $E_k$  intersect the real axis, while the phase  $F_k$  is special for the complex  $\beta$  case. By definition, the sets  $G_k$ ,  $F_k$ ,  $E_k$  are open.

Derrida [15], see also [25] for a rigorous proof, computed the limiting log-partition function of the REM at complex  $\beta$ . Namely, for the log-partition function of the REM corresponding to the  $k$ -th level of the GREM,

$$(2.5) \quad p_k(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n^{(k)}(\beta)|,$$

Derrida’s formula takes the form

$$(2.6) \quad p_k(\beta) = \begin{cases} |\sigma| \sqrt{2a_k \log \alpha_k}, & \text{if } \beta \in \bar{G}_k, \\ \frac{1}{2} \log \alpha_k + a_k \sigma^2, & \text{if } \beta \in \bar{F}_k, \\ \log \alpha_k + \frac{1}{2} a_k (\sigma^2 - \tau^2), & \text{if } \beta \in \bar{E}_k. \end{cases}$$

It is easy to check that the function  $p_k$  is continuous and strictly positive.

The next result shows that the limiting log-partition function of the GREM can be computed as the sum of the log-partition functions of the REM’s corresponding to the  $d$  levels of the GREM.

**Theorem 2.1.** *For every  $\beta \in \mathbb{C}$ , the following limit exists in probability and in  $L^q$ , for all  $q \geq 1$ :*

$$(2.7) \quad p(\beta) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \sum_{k=1}^d p_k(\beta),$$

where  $p_k(\beta)$ , the contribution of the  $k$ -th level, is given by (2.6).

**Remark 2.2.** Restricting (2.7) and (2.6) to the real temperature case  $\beta \geq 0$ , we obtain, for  $\beta \in [\sigma_m, \sigma_{m+1})$  with  $0 \leq m \leq d$ ,

$$p(\beta) = \sigma \sum_{k=1}^m \sqrt{2a_k \log \alpha_k} + \sum_{k=m+1}^d \left( \log \alpha_k + \frac{1}{2} a_k (\sigma^2 - \tau^2) \right).$$

Thus, we recovered the previously known formula obtained in [11]; see also [16, 7, 9].

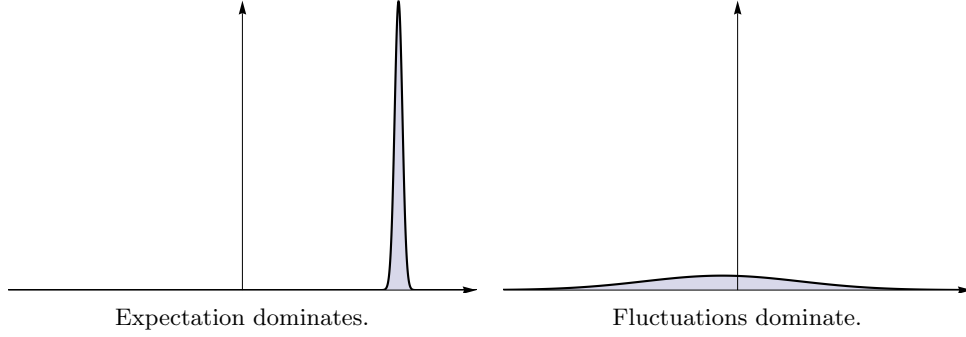


FIGURE 3. Caricatures of the probability density of  $\mathcal{Z}_n^{(k)}(\beta)$  in the regimes with light tails.

**2.2. Heuristics.** The reader may find the following heuristics useful. There are three natural guesses on the asymptotic behavior of  $\mathcal{Z}_n^{(k)}(\beta)$ :

- (a) *expectation dominates*:  $\mathcal{Z}_n^{(k)}(\beta)$  behaves approximately as its *expectation*; see Figure 3, left. This guess turns out to be correct in phase  $E_k$ .

However, it can happen that the fluctuations of  $\mathcal{Z}_n^{(k)}(\beta)$  around its expectation are of larger order than the expectation. In this case, we end up in the following regime:

- (b) *fluctuations dominate*:  $\mathcal{Z}_n^{(k)}(\beta)$  behaves approximately as its *standard deviation*; see Figure 3, right. This guess turns out to be correct in phase  $F_k$ .

Still, it can happen that due to the presence of heavy tails neither the expectation nor the standard deviation are adequate to estimate the true magnitude of the partition function. In this case, one can make the following guess:

- (c) *extremes dominate*:  $\mathcal{Z}_n^{(k)}(\beta)$  behaves approximately as the *maximal summand* in (2.1). This guess turns out to be correct in phase  $G_k$ .

Summarizing, we arrive at the following three guesses for the limiting log-partition function  $p_k(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n^{(k)}(\beta)|$ :

$$(2.8) \quad \text{Expectation} \quad p_k(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \mathbb{E} \mathcal{Z}_n^{(k)}(\beta) \right| = \log \alpha_k + \frac{1}{2} a_k (\sigma^2 - \tau^2),$$

$$(2.9) \quad \text{Fluctuations} \quad p_k(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sqrt{\text{Var} \mathcal{Z}_n^{(k)}(\beta)} = \frac{1}{2} \log \alpha_k + a_k \sigma^2,$$

$$(2.10) \quad \text{Extremes} \quad p_k(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \max_{j=1, \dots, N_{n,k}} \left| e^{\beta \sqrt{n a_k} \eta_j^{(k)}} \right| = |\sigma| \sqrt{2 a_k \log \alpha_k}.$$

It turns out that these formulae indeed give the correct value of  $p_k(\beta)$  in phases  $E_k$ ,  $F_k$ ,  $G_k$ , respectively.

**2.3. Global limiting distribution of complex zeros.** Using Theorem 2.1, it is possible to obtain the limiting distribution of complex zeros of the GREM partition function  $\mathcal{Z}_n(\beta)$ .

For  $\mathcal{Z}_n^{(k)}(\beta)$ , the partition function of the REM corresponding to the  $k$ -th level of the GREM, the limiting distribution of zeros has been computed by Derrida [15]; see also [25] for a rigorous proof. The main idea is to use the Poincaré–Lelong

formula (see, e.g., [20, §2.4.1]). It states that the measure counting the complex zeros of any analytic function  $f$  (which is not everywhere 0) can be represented as

$$(2.11) \quad \mathbf{Zeros}\{f(\beta): \beta \in \mathbb{C}\} = \frac{1}{2\pi} \Delta \log |f(\beta)|.$$

Here,  $\Delta = \frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \tau^2}$  is the Laplace operator in the complex  $\beta$ -plane. The Laplace operator should be understood in the sense of generalized functions (distributions). Applying this formula to  $f(\beta) = \mathcal{Z}_n^{(k)}(\beta)$ , dividing by  $n$ , interchanging the large  $n$  limit and the Laplacian (which should be justified), and using (2.5), one can show that weakly on  $\mathcal{M}(\mathbb{C})$ ,

$$\frac{1}{n} \mathbf{Zeros}\{\mathcal{Z}_n^{(k)}(\beta): \beta \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \frac{1}{2\pi} \Delta p_k.$$

The distributional Laplacian of  $p_k$  (see, e.g., Section 14.3 for the details of the computation), is a measure  $\Xi_k$  on  $\mathbb{C}$  given by

$$(2.12) \quad \Xi_k := \Delta p_k = \Xi_k^F + \Xi_k^{EF} + \Xi_k^{EG},$$

where  $\Xi_k^F, \Xi_k^{EF}, \Xi_k^{EG}$  are measures on the complex plane defined as follows:

- (a)  $\Xi_k^F$  is  $2a_k$  times the two-dimensional Lebesgue measure restricted to  $F_k$ .
- (b)  $\Xi_k^{EF}$  is  $\sqrt{a_k \log \alpha_k}$  times the one-dimensional length measure on the boundary between  $E_k$  and  $F_k$  (which consists of two circular arcs).
- (c)  $\Xi_k^{EG}$  is a measure having the density  $\sqrt{2}a_k|\tau|$  with respect to the one-dimensional length measure restricted to the boundary between  $E_k$  and  $G_k$  (which consists of four line segments).

Thus, the zeros of  $\mathcal{Z}_n^{(k)}(\beta)$  fill the *two-dimensional* region  $F_k$  asymptotically uniformly with density  $2a_k n$ , but some zeros concentrate around the boundary of  $E_k$  with *one-dimensional* density asymptotically proportional to  $n$ . The term  $\Xi_k^F$  is just the pointwise Laplacian of  $p_k$ , whereas the terms  $\Xi_k^{EF}$  and  $\Xi_k^{EG}$  appear because the normal derivative of the function  $p_k$  has a jump discontinuity on the boundary of the phase  $E_k$ . On the boundary between  $F_k$  and  $G_k$ , the normal derivative of  $p_k$  is continuous, hence this boundary makes no one-dimensional contribution to  $\Xi$ .

We now proceed to the complex zeros of  $\mathcal{Z}_n(\beta)$ , the partition function of the GREM. In view of Theorem 2.1, it is not surprising that the limiting distribution of zeros of  $\mathcal{Z}_n(\beta)$  can be obtained as a *superposition* of the limiting zeros distributions of the corresponding REM's.

**Theorem 2.3.** *The following convergence of random measures holds weakly on the space  $\mathcal{M}(\mathbb{C})$ :*

$$(2.13) \quad \frac{1}{n} \mathbf{Zeros}\{\mathcal{Z}_n(\beta): \beta \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \frac{1}{2\pi} \Xi,$$

where  $\Xi = \Delta p = \sum_{k=1}^d \Xi_k$ .

**2.4. Phase diagram.** We can now describe the phase diagram of the GREM in the complex  $\beta$  plane; see Figure 4. It is obtained as a superposition of the phase diagrams of the corresponding REM's. Take some  $\beta \in \mathbb{C}$ . For every  $k = 1, \dots, d$ , we can determine the phase ( $G_k$ ,  $F_k$ , or  $E_k$ ) to which  $\beta$  belongs and write the result in form of a sequence of length  $d$  over the alphabet  $\{G, F, E\}$ . However, it is easy

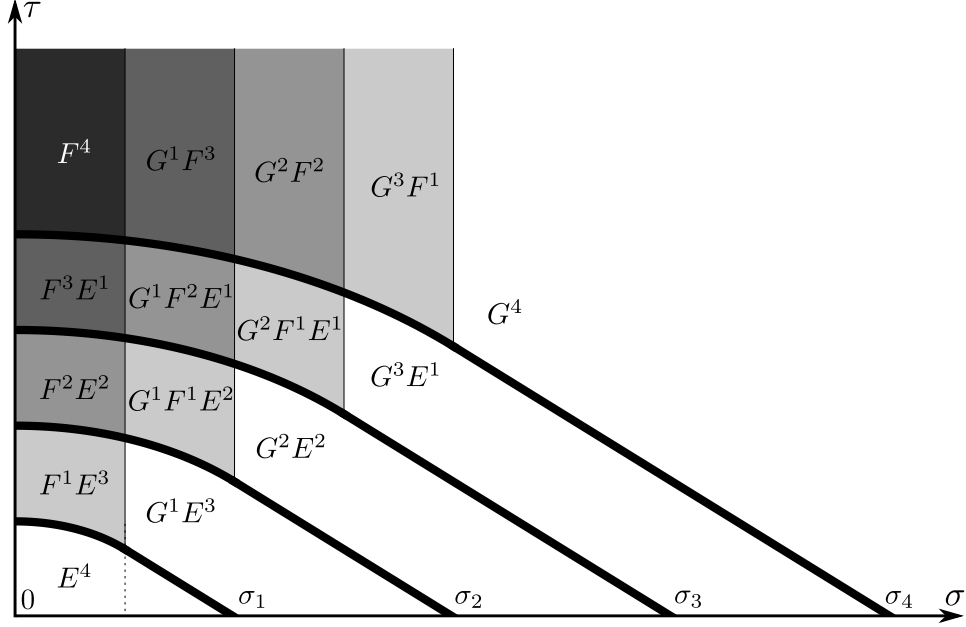


FIGURE 4. Phase diagram of a GREM with  $d = 4$  levels in the complex  $\beta$  plane. Only the quarter-plane  $\sigma \geq 0, \tau \geq 0$  is shown. Darker regions have larger density of partition function zeros.

to see that only phases of the following form are possible:

$$G^{d_1} F^{d_2} E^{d_3} = \underbrace{G \dots G}_{d_1} \underbrace{F \dots F}_{d_2} \underbrace{E \dots E}_{d_3},$$

where  $d_1, d_2, d_3 \in \{0, \dots, d\}$  are such that  $d_1 + d_2 + d_3 = d$ . In other words, we have an ordering of the level phases which can be symbolically expressed as

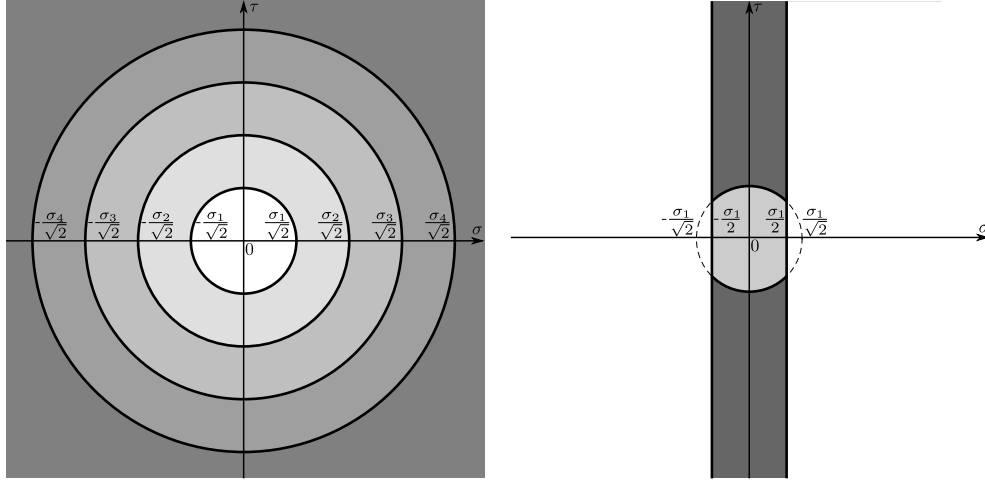
$$G \succ F \succ E.$$

For example, it is not possible that a level in  $E$ -phase is followed by a level in  $F$ - or in  $G$ -phase. This follows from the fact that if  $\beta \in E_k$  for some  $k$ , then  $\beta \notin F_l$  and  $\beta \notin G_l$  for  $l \geq k$ . This ordering of phases agrees with the observation of Saakian [40]. The phases of the GREM are therefore given by

$$G^{d_1} F^{d_2} E^{d_3} = (G_1 \cap \dots \cap G_{d_1}) \cap (F_{d_1+1} \cap \dots \cap F_{d_1+d_2}) \cap (E_{d_1+d_2+1} \cap \dots \cap E_d),$$

where  $d_1, d_2, d_3 \in \{0, \dots, d\}$  are such that  $d_1 + d_2 + d_3 = d$ . If  $\beta \in G^{d_1} F^{d_2} E^{d_3}$ , then we say that the levels  $1, \dots, d_1$  are in the  $G$ -phase, the levels  $d_1 + 1, \dots, d_1 + d_2$  are in the  $F$ -phase, and the levels  $d_1 + d_2 + 1, \dots, d$  are in the  $E$ -phase. Note that each  $G^{d_1} F^{d_2} E^{d_3}$  is an open subset of the complex plane. The union of the closures of these sets is the entire complex plane. The total number of phases is  $\frac{1}{2}(d+1)(d+2)$ . Only  $d+1$  of these phases, namely those of the form  $G^{d_1} E^{d_3}$ , intersect the real axis.

**2.5. Central limit theorem in the strip  $|\sigma| < \frac{\sigma_1}{2}$ .** In this and subsequent sections, we identify the limiting fluctuations of the partition function  $\mathcal{Z}_n(\beta)$ . We can view  $\mathcal{Z}_n(\beta)$  as a sum of random variables in a triangular summation scheme.



Regimes of the asymptotic behavior of  $\text{Var } \mathcal{Z}_n(\beta)$ ; see Proposition 2.6. Darker regions have stronger local correlations of  $\mathcal{Z}_n(\beta)$ ; see Section 6.

Two cases in the central limit theorem for  $\mathcal{Z}_n(\beta)$ . See Propositions 2.8 and 2.9.

FIGURE 5. Variance and CLT.

Although these random variables are dependent (unless  $d = 1$ ), the limiting distribution of their sum  $\mathcal{Z}_n(\beta)$  is infinitely divisible, as we shall see. It is well known that an infinitely divisible distribution can be decomposed into a superposition of a Gaussian and a Poissonian component. In this section, we consider the case in which only the Gaussian component is present. The next result states that in the strip  $|\sigma| < \frac{\sigma_1}{2}$  the partition function  $\mathcal{Z}_n(\beta)$  satisfies a central limit theorem.

**Theorem 2.4.** *Let  $\beta = \sigma + i\tau \in \mathbb{C} \setminus \{0\}$  be such that  $|\sigma| < \frac{\sigma_1}{2}$ . Then,*

$$(2.14) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \begin{cases} N_{\mathbb{C}}(0, 1), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0, 1), & \text{if } \tau = 0. \end{cases}$$

To draw corollaries from Theorem 2.4, we need to obtain expressions for  $\mathbb{E}\mathcal{Z}_n(\beta)$  and  $\text{Var } \mathcal{Z}_n(\beta)$ . Recall that  $a = a_1 + \dots + a_d$  denotes the variance of  $X_\varepsilon$ ,  $\varepsilon \in \mathbb{S}_n$ , and  $\alpha = \alpha_1 \cdot \dots \cdot \alpha_d$ . Recall the convention that  $\sigma_{d+1} = +\infty$ .

**Proposition 2.5.** *For every  $\beta \in \mathbb{C}$ ,  $\mathbb{E}\mathcal{Z}_n(\beta) = N_n e^{\frac{1}{2}\beta^2 a n}$ .*

*Proof.* If  $X \sim N_{\mathbb{R}}(0, \theta^2)$  is real normal random variable with mean zero and variance  $\theta^2$ , then  $\mathbb{E}e^{tX} = e^{\frac{1}{2}\theta^2 t^2}$ ,  $t \in \mathbb{C}$ . Since every Gaussian random variable  $X_\varepsilon$  in (1.7) has variance  $a$ , we immediately obtain the required formula.  $\square$

Next, we establish an asymptotic formula for  $\text{Var } \mathcal{Z}_n(\beta)$ , as  $n \rightarrow \infty$ . The asymptotic behavior of the variance displays several regimes (see Figure 5, left) which are separated by the circles

$$|\beta| = \frac{\sigma_k}{\sqrt{2}}, \quad 1 \leq k \leq d.$$

**Proposition 2.6.** *Let  $\beta \in \mathbb{C}$  be arbitrary. For  $0 \leq k \leq d$ , write*

$$b_k = \log \alpha + 2\sigma^2 a + \sum_{m=k+1}^d (\log \alpha_m - |\beta|^2 a_m).$$

*Then,*

$$\text{Var } \mathcal{Z}_n(\beta) \sim \begin{cases} e^{b_k n}, & \text{if } \frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}, \quad 1 \leq k \leq d, \\ e^{b_1 n}, & \text{if } 0 < |\beta| < \frac{\sigma_1}{\sqrt{2}}, \\ 2e^{b_k n}, & \text{if } |\beta| = \frac{\sigma_k}{\sqrt{2}}, \quad 2 \leq k \leq d, \\ e^{b_1 n}, & \text{if } |\beta| = \frac{\sigma_1}{\sqrt{2}}. \end{cases}$$

*In the first two cases, the formula holds locally uniformly as long as  $\beta$  stays in the specified region.*

As an immediate corollary of Proposition 2.6, we obtain the following result comparing the expectation and the standard deviation of  $\mathcal{Z}_n(\beta)$ .

**Proposition 2.7.** *For any  $\beta \in \mathbb{C} \setminus \{0\}$ ,*

$$\lim_{n \rightarrow \infty} \frac{|\mathbb{E} \mathcal{Z}_n(\beta)|}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} = \begin{cases} \infty, & \text{for } |\beta| < \frac{\sigma_1}{\sqrt{2}}, \\ 1, & \text{for } |\beta| = \frac{\sigma_1}{\sqrt{2}}, \\ 0, & \text{for } |\beta| > \frac{\sigma_1}{\sqrt{2}}. \end{cases}$$

Depending on which quantity, the expectation or the standard deviation, has larger order of magnitude, we can derive from Theorem 2.4 the following two corollaries. The corresponding domains are shown in Figure 5, right.

**Proposition 2.8.** *If  $|\beta| > \frac{\sigma_1}{\sqrt{2}}$  and  $|\sigma| < \frac{\sigma_1}{2}$  (which means that  $\beta \in F^{d_2} E^{d_3}$  with  $d_2 > 0$ ), then we can drop the expectation in (2.14):*

$$\frac{\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} N_{\mathbb{C}}(0, 1).$$

**Proposition 2.9.** *If  $|\beta| < \frac{\sigma_1}{\sqrt{2}}$  and  $|\sigma| < \frac{\sigma_1}{2}$  (which implies but is not equivalent to  $\beta \in E^d$ ), then*

$$\frac{\mathcal{Z}_n(\beta)}{\mathbb{E} \mathcal{Z}_n(\beta)} \xrightarrow[n \rightarrow \infty]{d} 1.$$

If  $\beta \in (-\frac{\sigma_1}{2}, +\frac{\sigma_1}{2})$  is real, then the result of Proposition 2.9 is contained in [7, Theorem 1.7]. Theorem 2.4 (which is stronger than Proposition 2.9) seems to be new even in the case  $\beta \in \mathbb{R}$ .

**2.6. Central limit theorem for  $|\sigma| = \frac{\sigma_1}{2}$ .** We will show that on the boundary of the strip, i.e. for  $|\sigma| = \frac{\sigma_1}{2}$ , the central limit theorem still holds, but with a non-standard limiting variance. In order to have the right “resolution” on the boundary, let us assume that  $\sigma = \sigma(n)$  depends on  $n$  in such a way that for some constant  $u \in \mathbb{R}$ ,

$$(2.15) \quad \sigma(n) = \frac{\sigma_1}{2} - \frac{u}{2\sqrt{na_1}} + o\left(\frac{1}{\sqrt{n}}\right).$$

**Theorem 2.10.** *Let  $\beta = \beta(n) = \sigma(n) + i\tau$  be such that  $\tau \in \mathbb{R}$  is constant and  $\sigma = \sigma(n)$  satisfies (2.15). Then,*

$$(2.16) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \begin{cases} N_{\mathbb{C}}(0, \Phi(u)), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0, \Phi(u)), & \text{if } \tau = 0. \end{cases}$$

Here,  $\Phi(u)$  is the standard normal distribution function.

In particular, if  $\sigma = \frac{\sigma_1}{2}$  does not depend on  $n$ , then  $u = 0$  and the variance of the limiting distribution is  $\frac{1}{2}$ . For the case of the REM and real  $\beta$ , this fact was discovered in [10]. See also [24] for a version with a fine “resolution” as in (2.15). For the case of the REM and complex  $\beta$ , see [25]. In the case of the GREM, Theorem 2.16 is new even in the real  $\beta$  case. The appearance of the “truncated variance” in (2.16) can be explained as follows. For  $\sigma < \frac{\sigma_1}{2}$ , the limiting distribution is Gaussian, whereas it turns out that for  $\sigma > \frac{\sigma_1}{2}$  the first level of the GREM contributes only to the Poissonian component of the limiting distribution. In the boundary case, some energies at the first level of the GREM have left the Gaussian part, but have not arrived yet at the Poissonian part. This is why the variance of the limiting Gaussian distribution is smaller than 1 in the boundary case.

**2.7. Poisson cascade zeta function.** The fluctuations of  $\mathcal{Z}_n(\beta)$  in phases of the form  $G^{d_1}F^{d_2}E^{d_3}$  with  $d_1 > 0$  will be described using a random zeta function associated to the Poisson cascades. In this section, we define this function and state results on its meromorphic continuation.

Let  $P_1, P_2, \dots$  be the points of a unit intensity Poisson point process on  $(0, \infty)$ . The points are always arranged in an increasing order. The *Poisson process zeta function* is defined by

$$\zeta_P(z) = \sum_{k=1}^{\infty} P_k^{-z}, \quad \text{Re } z > 1.$$

With probability 1, the above series converges absolutely and uniformly on compact subsets of the half-plane  $\{\text{Re } z > 1\}$  since  $\lim_{k \rightarrow \infty} P_k/k = 1$  a.s. by the law of large numbers. However, with probability 1, the function  $\zeta_P$  admits a meromorphic continuation to the half-plane  $\{\text{Re } z > 1/2\}$ . Namely, by [25, Theorem 2.6], with probability 1, we have

$$(2.17) \quad \sum_{P_k \leq T} P_k^{-z} - \int_1^T t^{-z} dt \xrightarrow[T \rightarrow \infty]{} \zeta_P(z) - \frac{1}{z-1} \text{ on } \mathcal{H}(\{\text{Re } z > 1/2\}).$$

We will need a multivariate generalization of the Poisson process zeta function which will be called the *Poisson cascade zeta function*. First, we need to define the Poisson cascade point processes; see Figure 6. These and related point processes appeared for example in [7], [39]. Fix dimension  $d \in \mathbb{N}$ . Start with a unit intensity Poisson point process  $\sum_{i=1}^{\infty} \delta(P_i)$  on  $(0, \infty)$ . Then, for every  $m = 1, \dots, d-1$  and every  $\varepsilon_1, \dots, \varepsilon_m \in \mathbb{N}$  let  $\sum_{i=1}^{\infty} \delta(P_{\varepsilon_1 \dots \varepsilon_m i})$  be a unit intensity Poisson point process on  $(0, \infty)$ . Assume that all point processes introduced above are independent. Consider the following point process  $\Pi$  on  $(0, \infty)^d$ ,

$$(2.18) \quad \Pi = \sum_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{N}^d} \delta(P_{\varepsilon_1}, P_{\varepsilon_1 \varepsilon_2}, \dots, P_{\varepsilon_1 \dots \varepsilon_d}).$$

Of course,  $\Pi$  is not a Poisson process (unless  $d = 1$ ) since  $\Pi$  contains infinitely many collinear point with probability 1. The next lemma states that  $\Pi$  has the

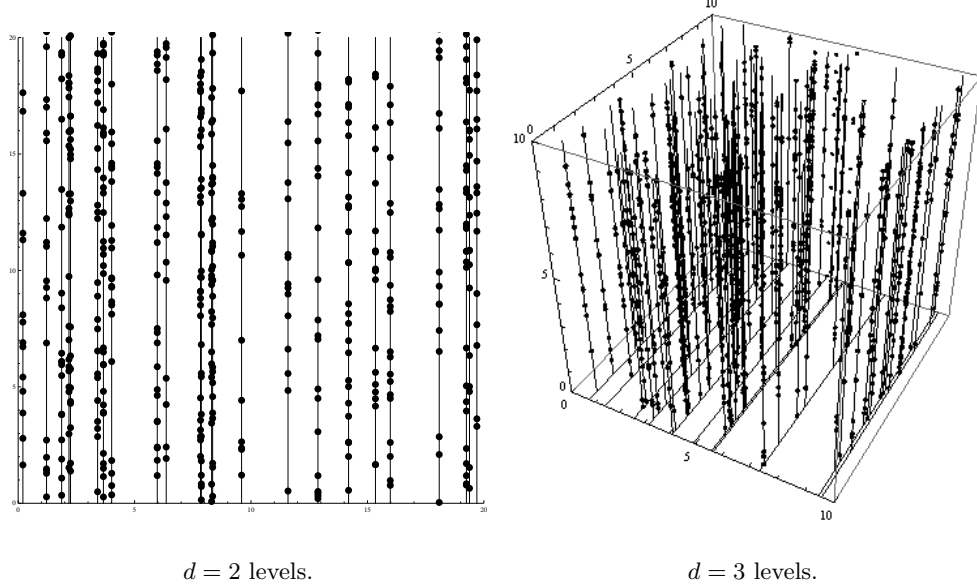


FIGURE 6. Poisson cascade point process.

same first order intensity as the homogeneous Poisson process on  $(0, \infty)^d$ . It can easily be proven by induction over  $d$ .

**Lemma 2.11.** *Let  $\varphi$  be an integrable or non-negative function on  $(0, \infty)^d$ . Then,*

$$\mathbb{E} \left[ \sum_{x \in \Pi} \varphi(x) \right] = \int_{(0, \infty)^d} \varphi(x) dx.$$

The random zeta function  $\zeta_P$  associated to the Poisson cascade point process  $\Pi$  is a stochastic process defined by the series

$$(2.19) \quad \zeta_P(z_1, \dots, z_d) = \sum_{\varepsilon \in \mathbb{N}^d} P_{\varepsilon_1}^{-z_1} P_{\varepsilon_1 \varepsilon_2}^{-z_2} \dots P_{\varepsilon_1 \dots \varepsilon_d}^{-z_d}.$$

**Theorem 2.12.** *With probability 1, the series (2.19) converges absolutely and uniformly on any compact subset of the domain*

$$(2.20) \quad \mathcal{D} = \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Re} z_1 > \dots > \operatorname{Re} z_d > 1\}.$$

*In particular, the function  $\zeta_P$  is analytic on  $\mathcal{D}$  with probability 1.*

Theorem 2.12 would be sufficient to treat the GREM at real inverse temperature  $\beta$ , as in [7]. However, for complex  $\beta$ , we need a meromorphic continuation of  $\zeta_P$  to a larger domain.

**Theorem 2.13.** *With probability 1, the function  $\zeta_P(z_1, \dots, z_d)$  defined originally on  $\mathcal{D}$  admits a meromorphic continuation to the domain*

$$\frac{1}{2}\mathcal{D} = \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Re} z_1 > \dots > \operatorname{Re} z_d > 1/2\}.$$

*Moreover, the function  $(z_d - 1)\zeta_P(z_1, \dots, z_d)$  is analytic on  $\frac{1}{2}\mathcal{D}$  with probability 1.*



We conjecture that with probability 1 there is no meromorphic continuation beyond  $\frac{1}{2}\mathcal{D}$ . In the sequel, we use the notation  $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ .

**Remark 2.14.** The value of  $(z_d - 1)\zeta_P(z)$  in the case  $z_d = 1$  is understood by continuity. In the case  $d = 1$ , this value is equal to 1, whereas, for  $d \geq 2$ , it is a non-degenerate random variable. (The non-degeneracy follows from the fact that a degenerate random variable cannot satisfy (2.21), see below, with  $\operatorname{Re} z_1 > z_d = 1$ ).

**Proposition 2.15.** Consider  $m \in \mathbb{N}$  independent copies of the random analytic function  $\{(z_d - 1)\zeta_P(z) : z \in \frac{1}{2}\mathcal{D}\}$  denoted by  $\{(z_d - 1)\zeta_P^{(j)}(z) : z \in \frac{1}{2}\mathcal{D}\}$ ,  $1 \leq j \leq m$ . Then, the following distributional equality on  $\mathcal{H}(\frac{1}{2}\mathcal{D})$  holds:

$$(2.21) \quad \left\{ \sum_{j=1}^m (z_d - 1)\zeta_P^{(j)}(z) : z \in \frac{1}{2}\mathcal{D} \right\} \stackrel{d}{=} \left\{ m^{z_1} (z_d - 1)\zeta_P(z) : z \in \frac{1}{2}\mathcal{D} \right\}.$$

From Proposition 2.15, we can draw several conclusions about the finite-dimensional distributions of  $\zeta_P$ . If  $z \in \frac{1}{2}\mathcal{D} \cap \mathbb{R}^d$ , then the distribution of the real-valued random variable  $(z_d - 1)\zeta_P(z)$  is *stable* with exponent  $1/z_1$ ; see [42, Chapter 1]. In fact, it is even *strictly stable* meaning that no additive constant is needed in (2.21). If  $z \in \frac{1}{2}\mathcal{D}$  is such that  $z_1 \in \mathbb{R}$  (but  $z_2, \dots, z_d$  are not necessarily real), then the term  $m^{z_1}$  is real and hence,  $(z_d - 1)\zeta_P(z)$  (which is considered as a random vector with values in  $\mathbb{C} \equiv \mathbb{R}^2$ ) has a two-dimensional stable distribution (which need not be isotropic); see [42, Chapter 2]. In general, for  $z \in \frac{1}{2}\mathcal{D}$  without any additional assumptions on the components, the distribution of the random variable  $(z_d - 1)\zeta_P(z)$  (again considered as a random vector with values in  $\mathbb{C} \equiv \mathbb{R}^2$ ) is *strictly complex stable* in the sense of Hudson and Veeh [21]. A random variable with values in  $\mathbb{C}$  is called strictly complex stable, see [21], if for every  $m \in \mathbb{N}$  the sum of  $m$  independent copies of this random variable, after dividing it by an appropriate complex number, has the same law as the original random variable. More generally, all finite-dimensional distributions of the stochastic process  $\{(z_d - 1)\zeta_P(z) : z \in \frac{1}{2}\mathcal{D}\}$  are *strictly operator stable* (and hence, infinitely divisible). Recall that a random vector with values in  $\mathbb{R}^k$  is called strictly operator stable, if for every  $m \in \mathbb{N}$  the sum of  $m$  copies of this random vector, after applying to it an appropriate linear transformation of  $\mathbb{R}^k$ , has the same law as the original random vector; see [32, Definition 3.3.24]. The same conclusions apply to the random variable  $\zeta_P(z)$  and the stochastic process  $\{\zeta_P(z) : z \in \frac{1}{2}\mathcal{D}\}$  if we additionally assume that  $z_d \neq 1$ .

The following property of the moments of  $\zeta_P(z)$  will be deduced in Section 8.5 from the operator stability.

**Proposition 2.16.** Let  $0 < p < 2$  and  $z \in \frac{1}{2}\mathcal{D}$ .

- (1) If  $\operatorname{Re} z_1 < \frac{1}{p}$ , then  $\mathbb{E}|(z_d - 1)\zeta_P(z)|^p < \infty$ .
- (2) If  $\operatorname{Re} z_1 > \frac{1}{p}$ , then  $\mathbb{E}|(z_d - 1)\zeta_P(z)|^p = \infty$  (unless  $d = 1$  and  $z = 1$ ).

**2.8. Fluctuations of the partition function.** First, we need to introduce several normalizing sequences. For each  $1 \leq k \leq d$ , let  $\{u_{n,k}\}_{n \in \mathbb{N}}$  be a real sequence satisfying

$$(2.22) \quad N_{n,k} \sim \sqrt{2\pi} u_{n,k} e^{\frac{1}{2} u_{n,k}^2}, \quad n \rightarrow \infty.$$

Equivalently, we can choose

$$(2.23) \quad u_{n,k} = \sqrt{2 \log N_{n,k}} - \frac{\log(4\pi \log N_{n,k}) + o(1)}{2\sqrt{2 \log N_{n,k}}} \sim \sqrt{2n \log \alpha_k} = \sigma_k \sqrt{n \alpha_k}.$$

It is well known, see [28, Theorem 1.5.3], that if  $\eta_1, \eta_2, \dots$  are independent real standard Gaussian random variables, then

$$u_{n,k} \left( \max_{i=1, \dots, N_{n,k}} \eta_i - u_{n,k} \right) \xrightarrow[n \rightarrow \infty]{d} e^{-e^{-x}}.$$

Let  $\beta \in \mathbb{C}$  be located inside (but not on the boundary) of some phase  $G^{d_1} F^{d_2} E^{d_3}$  and let  $\sigma \geq 0$ . For  $1 \leq k \leq d$ , we define a sequence of functions  $c_{n,k}(\beta)$  (which is needed to normalize the  $k$ -th level of the GREM) by

$$(2.24) \quad c_{n,k}(\beta) = \begin{cases} \beta \sqrt{n a_k} u_{n,k}, & \text{if } \beta \in G_k, \\ \frac{1}{2} \log N_{n,k} + a_k \sigma^2 n, & \text{if } \beta \in F_k, \\ \log N_{n,k} + \frac{1}{2} a_k \beta^2 n, & \text{if } \beta \in E_k. \end{cases}$$

Then, define a normalizing function  $c_n(\beta)$  by

$$(2.25) \quad c_n(\beta) = c_{n,1}(\beta) + \dots + c_{n,d}(\beta).$$

**Theorem 2.17.** *Let  $\beta \in G^{d_1} F^{d_2} E^{d_3}$  and let  $\sigma \geq 0$ . Then,*

$$\frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \begin{cases} 1, & \text{if } d_1 = 0 \text{ and } d_2 = 0, \\ N_{\mathbb{C}}(0, 1), & \text{if } d_1 = 0 \text{ and } d_2 > 0, \\ \zeta_P(\frac{\beta}{\sigma_1}, \dots, \frac{\beta}{\sigma_{d_1}}), & \text{if } d_1 > 0 \text{ and } d_2 = 0, \\ c S_{\sigma_1/\sigma}, & \text{if } d_1 > 0 \text{ and } d_2 > 0. \end{cases}$$

Here,  $\zeta_P$  is the Poisson cascade zeta function;  $S_\alpha$  is the rotationally symmetric, complex standard  $\alpha$ -stable random variable with characteristic function  $\mathbb{E} e^{i \operatorname{Re}(S_\alpha \bar{z})} = e^{-|z|^\alpha}$ ,  $z \in \mathbb{C}$ , where  $\alpha \in (0, 2)$ ; and  $c$  is a constant.

*Proof.* We will establish stronger results below. The case  $d_1 = 0, d_2 = 0$  follows from Theorem 2.19 below. The case  $d_1 = 0$  and  $d_2 > 0$  follows from Proposition 2.8. (For the asymptotics of the variance, see Proposition 2.6). The case  $d_1 > 0, d_2 = 0$  follows from Theorem 2.25 below. Finally, the case  $d_1 > 0, d_2 > 0$  follows from Theorem 2.28 (with  $t = 0$ ) below.  $\square$

**Remark 2.18.** The assumption  $\sigma \geq 0$  in Theorem 2.17 can be removed if we define

$$c_{n,k}(\beta) = (\operatorname{sgn} \sigma) \cdot \beta \sqrt{n a_k} u_{n,k} \text{ for } \beta \in G_k.$$

**2.9. Functional limit theorems and local structure of zeros.** One may ask whether the partition function  $\mathcal{Z}_n(\beta)$  converges, after an appropriate rescaling (involving, if necessary, a rescaling of the variable  $\beta$ ), to some limiting stochastic process. In this section, we state functional limit theorems of this type. Since weak convergence of random analytic functions implies weak convergence of point processes of zeros, see Proposition 3.13 below, any functional limit theorem implies a result on the local structure of zeros of  $\mathcal{Z}_n(\beta)$ .

**2.9.1. Phase  $E_1 = E^d$ .** The first result is a law of large numbers in the phase  $E_1 = E^d$ .

**Theorem 2.19.** *The following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_1)$ :*

$$(2.26) \quad \frac{\mathcal{Z}_n(\beta)}{\mathbb{E} \mathcal{Z}_n(\beta)} \xrightarrow[n \rightarrow \infty]{w} 1.$$

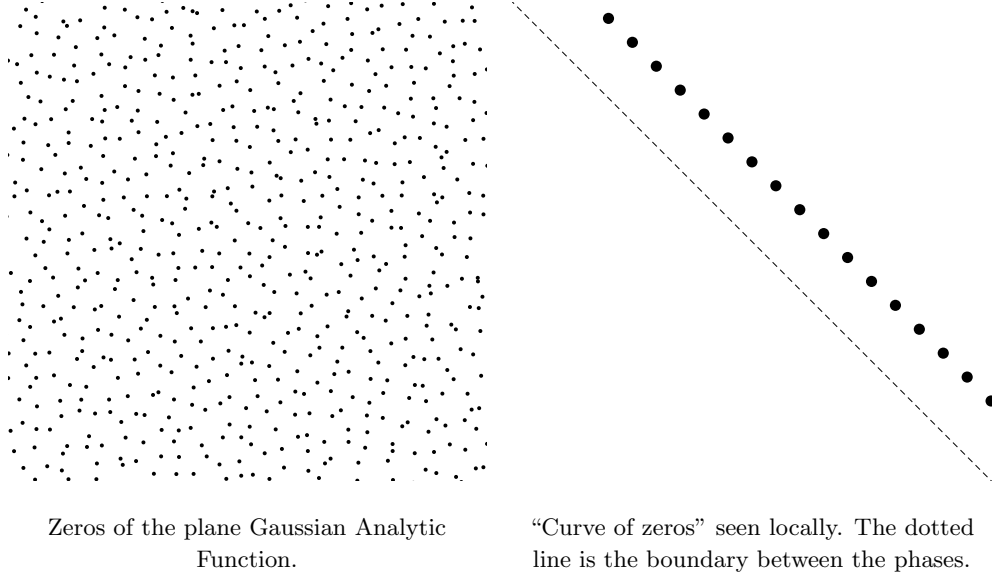


FIGURE 7. Point processes of zeros.

In the next two theorems, we will obtain more refined results by a “cleverer” choice of normalization. The first theorem deals with the domain  $E_1 \cap \{|\sigma| < \frac{\sigma_1}{2}\}$ . In this case, the limiting fluctuations of  $\mathcal{Z}_n(\beta)$  are given by the *plane Gaussian analytic function*  $\mathbb{X}$ ; see [20, 44]. It is a random analytic function  $\{\mathbb{X}(t) : t \in \mathbb{C}\}$  given by

$$(2.27) \quad \mathbb{X}(t) = e^{-\frac{t^2}{2}} \sum_{k=0}^{\infty} N_k \frac{t^k}{\sqrt{k!}},$$

where  $N_1, N_2, \dots \sim N_{\mathbb{C}}(0, 1)$  are independent complex standard Gaussian random variables. The finite-dimensional distributions of  $\mathbb{X}$  are multivariate complex Gaussian distributions and the second-order structure of  $\mathbb{X}$  is given by

$$(2.28) \quad \mathbb{E}\mathbb{X}(t) = 0, \quad \mathbb{E}[\mathbb{X}(t_1)\mathbb{X}(t_2)] = 0, \quad \mathbb{E}[\mathbb{X}(t_1)\overline{\mathbb{X}(t_2)}] = e^{-\frac{1}{2}(t_1 - \bar{t}_2)^2}, \quad t_1, t_2 \in \mathbb{C}.$$

The restriction of  $\mathbb{X}$  to  $\mathbb{R}$  is a stationary complex Gaussian process. The factor  $e^{-t^2/2}$  in (2.27) is chosen to simplify the statements of our results and is usually not used in the literature. The set of complex zeros of  $\mathbb{X}$  is a remarkable *stationary* point process; see Figure 7, left. The intensity of this point process is  $\pi^{-1}$ , that is for every Borel set  $B \subset \mathbb{C}$  we have

$$\mathbb{E} \left[ \sum_{z \in B} \mathbb{1}_{\mathbb{X}(z)=0} \right] = \frac{1}{\pi} \text{Leb}(B).$$

For more information on the zeros of  $\mathbb{X}$ , we refer to [20, 44].

We are ready to state the functional limit theorem in the domain  $E_1 \cap \{|\sigma| < \frac{\sigma_1}{2}\}$ . Recall the definition of  $c_{n,k}(\beta)$  from (2.24) and define

$$(2.29) \quad \tilde{c}_n(\beta) = c_{n,2}(\beta) + \dots + c_{n,d}(\beta).$$

**Theorem 2.20.** Fix  $\beta_* = \sigma_* + i\tau_* \in E_1 \cap \{|\sigma| < \frac{\sigma_1}{2}\}$ . Then, the following convergence of random analytic functions holds weakly on  $\mathcal{H}(\mathbb{C})$ :

$$(2.30) \quad \left\{ \frac{\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right) - \mathbb{E}\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right)}{N_{n,1}^{\frac{1}{2}} e^{a_1\left(\sigma_* + \frac{t}{\sqrt{n}}\right)^2} n e^{\tilde{c}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right)}} : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{\mathbb{X}(\sqrt{a_1}t) : t \in \mathbb{C}\},$$

where  $\{\mathbb{X}(t) : t \in \mathbb{C}\}$  is the plane Gaussian analytic function (2.27).

In the domain  $E_1 \cap \{\sigma > \frac{\sigma_1}{2}\}$ , the limiting fluctuations of  $\mathcal{Z}_n(\beta)$  are given by the Poisson zeta function.

**Theorem 2.21.** The following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_1 \cap \{\sigma > \frac{\sigma_1}{2}\})$ :

$$(2.31) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{\beta\sqrt{na_1}u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P\left(\frac{\beta}{\sigma_1}\right).$$

**Remark 2.22.** By symmetry, see (1.10), the following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_1 \cap \{\sigma < -\frac{\sigma_1}{2}\})$ :

$$(2.32) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{-\beta\sqrt{na_1}u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P^-\left(-\frac{\beta}{\sigma_1}\right),$$

where  $\zeta_P^-$  is a copy of  $\zeta_P$ . In fact, one can even show that the functional limit theorem holds on the *union* of both domains, namely  $E_1 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$ , and that the limiting functions  $\zeta_P$  and  $\zeta_P^-$  are *independent*; see Remark 11.2 and also Remark 2.26 for explanation.

It follows from the above results by an elementary calculation that in phase  $E_1$  the fluctuations of  $\mathcal{Z}_n(\beta)$  around its expectation are of smaller order than the expectation. One can therefore expect that the function  $\mathcal{Z}_n$  has no zeros in  $E_1$ . The next theorem makes this precise.

**Theorem 2.23.** Let  $K$  be a compact subset of  $E_1$ . Then, there exist  $C = C(K)$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $n \in \mathbb{N}$ ,

$$\mathbb{P}[\exists \beta \in K : \mathcal{Z}_n(\beta) = 0] < Ce^{-\varepsilon n}.$$

**Corollary 2.24.** The following weak convergence of point processes on  $\mathcal{N}(E_1)$  holds:

$$\mathbf{Zeros}\{\mathcal{Z}_n(\beta) : \beta \in E_1\} \xrightarrow[n \rightarrow \infty]{w} \emptyset.$$

Here,  $\emptyset$  denotes the empty point process on  $E_1$ .

**2.9.2. Phases of the form  $G^{d_1}E^{d_3}$ .** In the next theorem, we prove the functional convergence of the partition function  $\mathcal{Z}_n(\beta)$  in the phases of the form  $G^{d_1}E^{d_3}$ , where  $d_1, d_3 \in \{0, \dots, d\}$  satisfy  $d_1 + d_3 = d$ . The limiting process is given in terms of the  $d_1$ -variate Poisson cascade zeta function  $\zeta_P$ . Recall that  $c_n(\beta)$  was defined in (2.25). For  $1 \leq l \leq d$ , define

$$(2.33) \quad T^l(\beta) = \left(\frac{\beta}{\sigma_1}, \dots, \frac{\beta}{\sigma_l}\right) \in \mathbb{C}^l, \quad T^0(\beta) = \emptyset.$$

**Theorem 2.25.** Fix some  $d_1, d_3 \in \{0, \dots, d\}$  such that  $d_1 + d_3 = d$ . The following convergence of random analytic functions holds weakly on  $\mathcal{H}(G^{d_1}E^{d_3} \cap \{\sigma > 0\})$ :

$$(2.34) \quad \frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P(T^{d_1}(\beta)).$$

In particular, for  $d_1 = 0$ , the limiting process is  $\zeta_P(\emptyset) = 1$ , and we recover Theorem 2.19.

**Remark 2.26.** Let  $d_1 \geq 1$ . By symmetry, see (1.10), a result similar to Theorem 2.25 holds in the domain  $G^{d_1}E^{d_3} \cap \{\sigma < 0\}$ . Namely, the following convergence of random analytic functions holds weakly on  $\mathcal{H}(G^{d_1}E^{d_3} \cap \{\sigma < 0\})$ :

$$(2.35) \quad \frac{\mathcal{Z}_n(\beta)}{e^{c_n(-\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P^-(T^{d_1}(-\beta)),$$

where  $\zeta_P^-$  is a copy of  $\zeta_P$ . One can show that (2.34) and (2.35) can be combined into a *joint* convergence in the phase  $G^{d_1}E^{d_3}$  and that the limiting functions  $\zeta_P$  and  $\zeta_P^-$  are *independent*, for  $d_1 \geq 1$ . We will not provide a complete proof of the independence, but let us explain the idea. The function  $\zeta_P$  in (2.34) appears as the contribution of the *upper* extremal order statistics among the energies on the first  $d_1$  levels of the GREM, whereas all other levels make a deterministic contribution equal to the expectation. The function  $\zeta_P^-$  in (2.35) appears as the contribution of the *lower* extremal order statistics on the first  $d_1$  levels of the GREM. Since upper and lower extremal order statistics become independent in the large sample limit, the limiting functions  $\zeta_P$  and  $\zeta_P^-$  are independent.

**Corollary 2.27.** *Under the same conditions as in Theorem 2.25, the following weak convergence of point processes on  $\mathcal{N}(G^{d_1}E^{d_3} \cap \{\sigma > 0\})$  holds:*

$$\mathbf{Zeros}\{\mathcal{Z}_n(\beta)\} \xrightarrow[n \rightarrow \infty]{w} \mathbf{Zeros}\{\zeta_P(T^{d_1}(\beta))\}.$$

Note that the intensity of zeros in the limiting point process is  $O(1)$  and hence these zeros do not appear in the limit in Theorem 2.3. For  $d_1 = 0$ , the limiting point process of zeros is empty and we recover Corollary 2.24.

**2.9.3. Phases with at least one fluctuation level.** Our next result is a functional limit theorem describing the limiting structure of the stochastic process  $\mathcal{Z}_n(\beta)$  in an infinitesimal neighborhood of some  $\beta_* = \sigma_* + i\tau_* \in G^{d_1}F^{d_2}E^{d_3}$ , where  $d_2 \geq 1$ .

**Theorem 2.28.** *Fix some  $d_1, d_2, d_3 \in \{0, \dots, d\}$  with  $d_1 + d_2 + d_3 = d$  and  $d_2 \geq 1$ . Also, fix some  $\beta_* = \sigma_* + i\tau_* \in G^{d_1}F^{d_2}E^{d_3}$  such that  $\sigma_* \geq 0$ . Then, for a suitable normalizing function  $c_n(\beta_*; t)$  (which is quadratic in  $t$ ), the following convergence of random analytic functions holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$\left\{ e^{-c_n(\beta_*; t)} \mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ \sqrt{W} \mathbb{X}(\kappa t) : t \in \mathbb{C} \right\},$$

where

- (1)  $W = \zeta_P(2T^{d_1}(\sigma_*))$  and  $\zeta_P$  is the Poisson cascade zeta function with  $d_1$  variables;
- (2)  $\{\mathbb{X}(t) : t \in \mathbb{C}\}$  is the plane Gaussian analytic function (2.27);
- (3)  $\kappa^2 = \sum_{k=1}^{d_2} a_{d_1+k}$  is the total variance of the GREM levels which are in the fluctuation phase;
- (4) the processes  $\zeta_P$  and  $\mathbb{X}$  are independent.

The formula for  $c_n(\beta_*; t)$  will be given in (13.3), (13.4) below. If  $d_1 = 0$  (i.e., there are no levels in the glassy phase), then the limit is the Gaussian analytic function  $\mathbb{X}(\kappa t)$  since we have  $\zeta_P(\emptyset) = 1$ . In the case  $d_1 \neq 0$ , the limiting process is a Gaussian process rescaled by the square root of an independent real  $\frac{\sigma_1}{2\sigma_*}$ -stable

random variable  $W = \zeta_P(2T^{d_1}(\sigma_*))$  with skewness parameter  $+1$ . Such a process is itself complex  $\frac{\sigma_1}{\sigma_*}$ -stable with complex isotropic margins. Processes of this type are called subgaussian; see [42].

**Corollary 2.29.** *Under the same conditions as in Theorem 2.28, the following convergence of the point processes of zeros holds weakly on  $\mathcal{N}(\mathbb{C})$ :*

$$\mathbf{Zeros} \left\{ \mathcal{Z}_n \left( \beta_* + \frac{t}{\sqrt{n}} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \mathbf{Zeros} \{ \mathbb{X}(\kappa t) : t \in \mathbb{C} \}.$$

**2.9.4. Curves of zeros: Beak shaped boundaries.** In phase  $G^{l-1}E^{d-l+1}$  the fluctuations of  $\mathcal{Z}_n(\beta)$  are given by an  $(l-1)$ -variate Poisson cascade zeta function, whereas in phase  $G^lE^{d-l}$  the fluctuations are given by an  $l$ -variate zeta function. On the boundary between these two phases, under an appropriate scaling, *both* functions become visible in the limit.

**Theorem 2.30.** *Fix some  $1 \leq l \leq d$  and some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  such that  $\sigma_* > \frac{\sigma_1}{2}$ ,  $\tau_* > 0$  and  $\sigma_* + \tau_* = \sigma_l$ . Then, there exist a complex sequence  $d_{n,l} = O(\log n)$  and a sequence of functions  $h_{n,l}(t)$  (which are quadratic functions in  $t$ ) such that weakly on  $\mathcal{H}(\mathbb{C})$  it holds that*

$$\left\{ e^{-h_{n,l}(t)} \mathcal{Z}_n \left( \beta_* + \frac{d_{n,l} + t}{a_l(\beta_* - \sigma_l)n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ e^t \zeta^{(l-1)} + \zeta^{(l)} : t \in \mathbb{C} \right\}.$$

Here,  $(\zeta^{(l-1)}, \zeta^{(l)})$  is a random vector given by

$$(2.36) \quad (\zeta^{(l-1)}, \zeta^{(l)}) = (\zeta_P(T^{l-1}(\beta_*)), \zeta_P(T^l(\beta_*))).$$

In (2.36), both  $\zeta^{(l-1)}$  and  $\zeta^{(l)}$  are based on the same Poisson cascade point process.

We will provide exact expressions for  $d_{n,l}$  and  $h_{n,l}(t)$  in (12.2) and (12.4), below. Theorem 2.30 allows us to clarify the local structure of the line of zeros on the beak shaped boundary between the phases  $G^{l-1}E^{d-l+1}$  and  $G^lE^{d-l}$ .

**Corollary 2.31.** *Under the same conditions as in Theorem 2.30, the following convergence of point processes holds weakly on  $\mathcal{N}(\mathbb{C})$ :*

$$\mathbf{Zeros} \left\{ \mathcal{Z}_n \left( \beta_* + \frac{d_{n,l} + t}{a_l(\beta_* - \sigma_l)n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \sum_{k \in \mathbb{Z}} \delta \left( \log \left( -\frac{\zeta^{(l-1)}}{\zeta^{(l)}} \right) + 2\pi i k \right).$$

It follows that the zeros of  $\mathcal{Z}_n(\beta)$  in a neighborhood of  $\beta_*$  look locally like *equally spaced* points on a line parallel to the boundary between  $G^{l-1}E^{d-l+1}$  and  $G^lE^{d-l}$ ; see Figure 7, right. The spacing between neighboring zeros is

$$\frac{\sqrt{2\pi}}{a_l \tau_*} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right).$$

This agrees with what one expects from the definition of the measure  $\Xi_l^{EG}$ ; see Section 2.3. From the formula for  $d_{n,l}$ , see (12.2), it can be seen that the zeros are located *outside* the phase  $E_l$ , the distance to the boundary being of order  $\text{const} \cdot \frac{\log n}{n}$ .

**2.9.5. Curves of zeros: Arc shaped boundaries.** In the next theorem, we describe the local structure of the partition function  $\mathcal{Z}_n(\beta)$  in an infinitesimal neighborhood of some  $\beta_* = \sigma_* + i\tau_*$  located on the boundary separating the phases  $G^{d_1} F^{d_2} E^{d_3}$  and  $G^{d_1} F^{d_2-1} E^{d_3+1}$ , where  $d_2 \geq 1$ . We assume that

$$(2.37) \quad \frac{\sigma_{d_1}}{2} < \sigma_* < \frac{\sigma_{d_1+1}}{2}, \quad \tau_* > 0, \quad \sigma_*^2 + \tau_*^2 = \frac{\sigma_{d_1+d_2}^2}{2}.$$

**Theorem 2.32.** *Fix some  $d_1, d_2, d_3 \in \{0, \dots, d\}$  with  $d_1 + d_2 + d_3 = d$  and  $d_2 \geq 2$ . Also, fix some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  satisfying (2.37). Then, for suitable normalizing functions  $f_n(\beta_*; t)$  (which are linear in  $t$ ), the following convergence of random analytic functions holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$\left\{ e^{-f_n(\beta_*; t)} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ \sqrt{W} (e^{\lambda' t} N' + e^{\lambda'' t} N'') : t \in \mathbb{C} \right\},$$

where

- (1)  $W = \zeta_P(2T^{d_1}(\sigma_*))$  and  $\zeta_P$  is the Poisson cascade zeta function with  $d_1$  variables;
- (2)  $N', N'' \sim N_{\mathbb{C}}(0, 1)$  are independent complex standard normal random variables;
- (3)  $\lambda', \lambda''$  are constants given in Remark 2.33 below;
- (4) the random variable  $W$  and the random vector  $(N', N'')$  are independent.

**Remark 2.33.** Define the “partial variances”  $A_{l,m} = a_l + \dots + a_m$  for  $1 \leq l \leq m \leq d$  and let  $A_{l,m} = 0$  if  $l > m$ . The constants  $\lambda'$  and  $\lambda''$  are given by

$$\begin{aligned} \lambda' &= 2\sigma_* A_{d_1+1, d_1+d_2} + \beta_* A_{d_1+d_2+1, d}, \\ \lambda'' &= 2\sigma_* A_{d_1+1, d_1+d_2-1} + \beta_* A_{d_1+d_2, d}. \end{aligned}$$

Note that  $\lambda' - \lambda'' = \bar{\beta} a_{d_1+d_2}$ . A formula for the normalizing function  $f_n(\beta_*; t)$  will be provided in (13.20) below.

**Corollary 2.34.** *Under the same conditions as in Theorem 2.32, we have the following weak convergence of point processes on  $\mathcal{N}(\mathbb{C})$ :*

$$\text{Zeros} \left\{ \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \sum_{k \in \mathbb{Z}} \delta \left( \frac{1}{\beta a_{d_1+d_2}} \left( \log \left( -\frac{N''}{N'} \right) + 2\pi i k \right) \right).$$

In the case  $d_2 = 1$  we have a slightly different result.

**Theorem 2.35.** *Fix some  $d_1, d_2, d_3 \in \{0, \dots, d\}$  with  $d_1 + d_2 + d_3 = d$  and  $d_2 = 1$ . Also, fix some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  satisfying (2.37). Then, for suitable normalizing functions  $f_n(\beta_*; t)$  (which are linear in  $t$ ), the following convergence of random analytic functions holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$\left\{ e^{-f_n(\beta_*; t)} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ e^{\lambda' t} \sqrt{W} N + e^{\lambda'' t} \zeta^{(d_1)} : t \in \mathbb{C} \right\},$$

where

- (1)  $W = \zeta_P(2T^{d_1}(\sigma_*))$  and  $\zeta^{(d_1)} = \zeta_P(T^{d_1}(\beta_*))$ , where in both cases the zeta function  $\zeta_P$  is based on the same Poisson cascade point process;
- (2)  $N \sim N_{\mathbb{C}}(0, 1)$  is a complex standard normal random variable;
- (3)  $\lambda'$  and  $\lambda''$  are constants given in Remark 2.33;
- (4) the random vector  $(W, \zeta^{(d_1)})$  and the random variable  $N$  are independent.

An explicit formula for  $f_n(\beta_*; t)$  will be given in (13.20) below.

**Corollary 2.36.** *Under the same conditions as in Theorem 2.35, we have the following weak convergence of point processes on  $\mathcal{N}(\mathbb{C})$ :*

$$\mathbf{Zeros} \left\{ \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : \beta \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \sum_{k \in \mathbb{Z}} \delta \left( \frac{1}{\beta_* a_{d_1+1}} \left( \log \left( -\frac{\zeta^{(d_1)}}{\sqrt{WN}} \right) + 2\pi i k \right) \right).$$

Both in Corollary 2.34 and Corollary 2.36 the zeros of  $\mathcal{Z}_n(\beta)$  in a neighborhood of  $\beta_*$  look locally like equally spaced points, the spacing being

$$\frac{2\pi}{a_{d_1+d_2} |\beta_*|} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right).$$

This agrees with what we expect from the definition of the measure  $\Xi_{d_1+d_2}^{EF}$ ; see Section 2.3.

**2.9.6. Fluctuations on the vertical half-line boundaries.** Let us finally state a theorem on the fluctuations of  $\mathcal{Z}_n(\beta)$  for  $\beta$  on the boundary between  $F_l$  and  $G_l$ , for some  $1 \leq l \leq d$ . This theorem is obtained by adjoining  $l-1$  glassy phase levels to Theorem 2.10.

**Theorem 2.37.** *Let  $\beta \in \mathbb{C}$  be such that  $\sigma = \frac{\sigma_l}{2}$  and  $\tau > \frac{\sigma_l}{2}$ , for some  $1 \leq l \leq d$ . Then, for a suitable normalizing sequence  $r_n(\beta)$ ,*

$$\frac{\mathcal{Z}_n(\beta)}{e^{r_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \frac{1}{\sqrt{2}} \sqrt{\zeta_P(2T^{l-1}(\sigma))} N,$$

where  $N \sim N_{\mathbb{C}}(0, 1)$  is independent of  $\zeta_P$ .

An exact expression for  $r_n(\beta)$  will be provided in (13.22) below. At the “triple points” (i.e., at points, where the phases  $E_l$ ,  $F_l$  and  $G_l$  meet), the result takes the following form.

**Theorem 2.38.** *Let  $\beta \in \mathbb{C}$  be such that  $\sigma = \tau = \frac{\sigma_l}{2}$ , for some  $1 \leq l \leq d$ . Then, for a suitable normalizing sequence  $r_n(\beta)$ ,*

$$\frac{\mathcal{Z}_n(\beta)}{e^{r_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \frac{1}{\sqrt{2}} \left( \sqrt{\zeta_P(2T^{l-1}(\sigma))} N + \zeta_P(T^{l-1}(\beta)) \right),$$

where  $N \sim N_{\mathbb{C}}(0, 1)$  is independent from the zeta functions and both zeta functions are based on the same Poisson cascade point process.

**2.10. Passing to continuum hierarchies.** In the GREM with  $d$  levels, there are  $d$  (real temperature) phase transitions at inverse temperatures  $\beta = \sigma_1, \dots, \sigma_d$ , whereas more interesting spin glass models like the Sherrington–Kirkpatrick model are known (or conjectured) to exhibit a “continuum of freezing phase transitions” or the so-called *full replica symmetry breaking*. It has been suggested by Derrida and Gardner [16] that it is possible to consider the limit of the GREM as  $d$ , the number of levels, goes to  $\infty$ . Bovier and Kurkova [8] defined the continuum limit of the GREM, the Continuous Random Energy Model (CREM), and computed its free energy at real  $\beta$ . In this section, we will show heuristically how to pass to the continuum hierarchy limit of the GREM in the complex  $\beta$  case; see Figure 8. It should be stressed that the arguments in this section are not entirely rigorous.



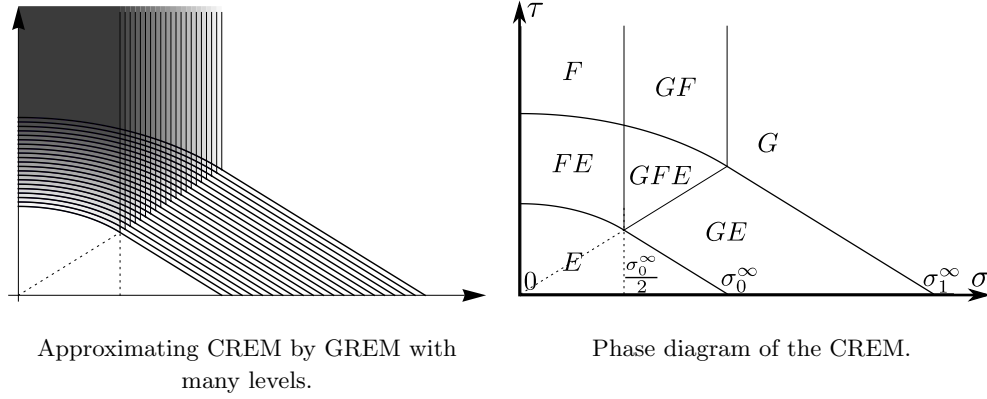


FIGURE 8. Phase diagram of the CREM.

Let  $A: [0, 1] \rightarrow \mathbb{R}$  be an increasing, concave function with  $A(0) = 0$ . Fix also some  $\alpha > 1$ . Consider a GREM with  $d$  levels whose parameters  $(a_1, \dots, a_d)$  and  $(\alpha_1, \dots, \alpha_d)$  are given by

$$(2.38) \quad a_1 + \dots + a_k = A\left(\frac{k}{d}\right), \quad \log \alpha_k = \frac{1}{d} \log \alpha, \quad 1 \leq k \leq d.$$

The total number of energies in this GREM is  $\alpha^{n+o(1)}$  and the variance of each energy is  $A(1)n$ .

Let us now pass to the large  $d$  limit. Let  $t \in [0, 1]$ . Then, it follows from (2.38) that the large  $d$  limit of  $da_{[td]}$  is  $A'(t)$ . Hence, the large  $d$  limit of the critical temperature  $\sigma_{[td]}$  is

$$\sigma_t^\infty = \sqrt{\frac{2 \log \alpha}{A'(t)}}.$$

The large  $d$  limits of the domains  $G_{[td]}$ ,  $F_{[td]}$ ,  $E_{[td]}$  are the domains

$$\begin{aligned} G_t^\infty &= \{\beta \in \mathbb{C} : 2|\sigma| > \sigma_t^\infty, |\sigma| + |\tau| > \sigma_t^\infty\}, \\ F_t^\infty &= \{\beta \in \mathbb{C} : 2|\sigma| < \sigma_t^\infty, 2(\sigma^2 + \tau^2) > (\sigma_t^\infty)^2\}, \\ E_t^\infty &= \mathbb{C} \setminus \overline{G_t^\infty \cup F_t^\infty}. \end{aligned}$$

Recall that the complex plane phases of a GREM with  $d$  levels were denoted by  $G^{d_1} F^{d_2} E^{d_3}$ , where the parameters  $d_1, d_2, d_3 \in \mathbb{N}_0$  satisfy  $d_1 + d_2 + d_3 = d$ . Instead of  $d_1, d_2, d_3$ , in the continuum limit we have three parameters  $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$  which are the large  $d$  limits of  $\frac{d_1}{d}, \frac{d_2}{d}, \frac{d_3}{d}$  and hence satisfy  $\gamma_1 + \gamma_2 + \gamma_3 = 1$ . To find the formula for  $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$  note that in the  $d$ -level GREM,

$$d_1 = \max\{k \geq 0 : \beta \in G_k\}, \quad d_1 + d_2 + 1 = \max\{k \geq 0 : \beta \in E_k\}.$$

Passing to the large  $d$  limit, we obtain

$$\gamma_1 = \sup\{t \in [0, 1] : \beta \in G_t^\infty\}, \quad \gamma_1 + \gamma_2 = \sup\{t \in [0, 1] : \beta \in E_t^\infty\}.$$

For the  $d$ -level GREM, Theorem 2.1 states that the log-partition function is  $p(\beta) = p_G(\beta) + p_F(\beta) + p_E(\beta)$ , where  $p_G(\beta)$ ,  $p_F(\beta)$ ,  $p_E(\beta)$  are the contributions of spin

glass, fluctuation and expectation levels given by

$$(2.39) \quad p_G(\beta) = |\sigma| \sum_{k=1}^{d_1} \sqrt{2a_k \log \alpha_k},$$

$$(2.40) \quad p_F(\beta) = \sum_{k=d_1+1}^{d_1+d_2} \left( \frac{1}{2} \log \alpha_k + a_k \sigma^2 \right),$$

$$(2.41) \quad p_E(\beta) = \sum_{k=d_1+d_2+1}^d \left( \log \alpha_k + \frac{1}{2} a_k (\sigma^2 - \tau^2) \right).$$

Replacing Riemann sums by Riemann integrals, we obtain that in the large  $d$  limit, the log-partition function is

$$(2.42) \quad p^\infty(\beta) = p_G^\infty(\beta) + p_F^\infty(\beta) + p_E^\infty(\beta),$$

where

$$(2.43) \quad p_G^\infty(\beta) = |\sigma| \sqrt{2 \log \alpha} \int_0^{\gamma_1} \sqrt{A'(t)} dt,$$

$$(2.44) \quad p_F^\infty(\beta) = \frac{\gamma_2}{2} \log \alpha + (A(\gamma_1 + \gamma_2) - A(\gamma_1)) \sigma^2,$$

$$(2.45) \quad p_E^\infty(\beta) = \gamma_3 \log \alpha + \frac{1}{2} (\sigma^2 - \tau^2) (A(1) - A(\gamma_1 + \gamma_2)).$$

If  $\beta$  is real, then  $\gamma_1$  is the solution of  $\sigma_{\gamma_1}^\infty = \sigma$ ,  $\gamma_2 = 0$ ,  $\gamma_3 = 1 - \gamma_1$  and the log-partition function is given by

$$(2.46) \quad p^\infty(\beta) = |\sigma| \sqrt{2 \log \alpha} \int_0^{\gamma_1} \sqrt{A'(t)} dt + (1 - \gamma_1) \log \alpha + \frac{\sigma^2}{2} (A(1) - A(\gamma_1)).$$

This formula is known, see [8, Theorem 3.3] (where the second term seems to be missing) and [9, Theorem 4.2] (where all terms are present).

In the continuum limit of the GREM, there are seven phases which we denote by

$$GFE, GF, FE, GE, G, F, E;$$

see Figure 8. In such a phase, for every letter which is not in the name of this phase, the corresponding  $\gamma_i$  must vanish. For example, the phase  $FE$  is characterized by the conditions  $\gamma_1 = 0$ ,  $\gamma_2 \neq 0$ ,  $\gamma_3 \neq 0$ .

It should be stressed that we do not have a rigorous proof that (2.42), (2.43), (2.44), (2.45) apply to the CREM as defined in [8]. In the real  $\beta$  case, Bovier and Kurkova [8] use were able to sandwich a CREM between two close GREM's which allowed them to derive (2.46) rigorously using Gaussian comparison inequalities. This method does not seem to work in the complex  $\beta$  case because we cannot apply the comparison inequalities.

The Branching Random Walk and the Gaussian Multiplicative Chaos can be seen as special (or limiting) cases of the CREM with  $A(t) = t$ . In this case,  $\sigma_t^\infty \equiv 1$  which means that we have only the phases  $E, F, G$  as in the REM, see [18, 27, 30, 31].

**2.11. Further extensions of the model.** Similarly to the setup of [25, Section 2.3], one can consider a complex GREM with arbitrary correlations between the real and imaginary parts of the random exponents. That is, given correlation

parameters  $\rho_1, \dots, \rho_d \in [-1, 1]$ , consider a Gaussian random field  $\{Y_\varepsilon : \varepsilon \in \mathbb{S}_n\}$  having the same distribution as  $\{X_\varepsilon : \varepsilon \in \mathbb{S}_n\}$ , see (1.5), and satisfying

$$(2.47) \quad \text{Cov}(X_\varepsilon, Y_\eta) = \sum_{k=1}^{l(\varepsilon, \eta)} \rho_k a_k, \quad \varepsilon, \eta \in \mathbb{S}_n,$$

where  $l(\varepsilon, \eta) = \min\{k \in \mathbb{N} : \varepsilon_k \neq \eta_k\} - 1$ . Along the lines of the present paper, one can study the partition function

$$(2.48) \quad \hat{\mathcal{Z}}_n(\beta) = \sum_{\varepsilon \in \mathbb{S}_n} e^{\sqrt{n}(\sigma X_\varepsilon + i\tau Y_\varepsilon)}, \quad \beta = (\sigma, \tau) \in \mathbb{R}^2.$$

It seems that Theorems 2.1 and 2.3 need no changes even if we substitute partition function (1.7) with the one from (2.48). The more refined results on fluctuations such as Theorem 2.17, however, need appropriate modifications; see [25] for the case  $d = 1$ .

**2.12. Structure of the proofs.** The remaining part of the paper is devoted to proofs. In order to obtain the fluctuations of  $\mathcal{Z}_n(\beta)$ , we will use the following approach. We will write the partition function  $\mathcal{Z}_n(\beta)$  as

$$\mathcal{Z}_n(\beta) = \sum_{k=1}^{N_{n,1}} e^{\beta \sqrt{na_1} \xi_k} \tilde{\mathcal{Z}}_{n,k}(\beta),$$

where  $e^{\beta \sqrt{na_1} \xi_k}$  are the contributions of the first GREM level, and  $\tilde{\mathcal{Z}}_{n,k}(\beta)$  are the contributions of the remaining  $d - 1$  levels which are given by

$$\tilde{\mathcal{Z}}_{n,k}(\beta) = \sum_{\varepsilon_2=1}^{N_{n,2}} \dots \sum_{\varepsilon_d=1}^{N_{n,d}} e^{\beta \sqrt{n}(\sqrt{a_2} \xi_{k\varepsilon_2} + \dots + \sqrt{a_d} \xi_{k\varepsilon_2 \dots \varepsilon_d})}.$$

This provides a representation of  $\mathcal{Z}_n(\beta)$  as a sum of independent random variables (in a triangular scheme), and the powerful theory of summation of independent random variables and vectors [35, 32] can be used. A similar approach was used in the case of the REM at real temperature by Bovier et al. [10]. The main question is what are the properties of random variables  $e^{\beta \sqrt{na_1} \xi_k}$  and  $\tilde{\mathcal{Z}}_{n,k}(\beta)$ . Depending on the behavior of the contributions of the first level, we will distinguish between two cases: the *Gaussian case* and the *Poissonian case*.

**GAUSSIAN CASE:**  $|\sigma| < \frac{\sigma_1}{2}$ . The main feature of this case is that it is possible to verify the *Lindeberg condition* for the random variables  $e^{\beta \sqrt{na_1} \xi_k}$ . As a consequence, the fluctuations of  $\mathcal{Z}_n(\beta)$  turn out to be Gaussian. The Gaussian case includes the sets  $F^{d_2} E^{d-d_2}$  with  $1 \leq d_2 \leq d$ , the set  $E_1 \cap \{|\sigma| < \frac{\sigma_1}{2}\}$ , and the boundaries between these sets. Note that only a part of the phase  $E_1 = E^d$  is included in the Gaussian case. The Gaussian case will be analyzed in Sections 4, 6, 7. In Section 5, we analyze the case  $|\sigma| = \frac{\sigma_1}{2}$ . Although the Lindeberg condition is not satisfied in this case, we will verify some weaker conditions which ensure that the limiting fluctuations of  $\mathcal{Z}_n(\beta)$  are still Gaussian.

**POISSONIAN CASE:**  $|\sigma| > \frac{\sigma_1}{2}$ . In this case, the random variables  $e^{\beta \sqrt{na_1} \xi_k}$  do not satisfy the Lindeberg condition. Instead, it turns out that the contribution of the first level comes from the *extremal order statistics* among the energies on the first level. The limiting fluctuations of  $\mathcal{Z}_n(\beta)$  in the Poissonian case will be described in terms of a Poisson cascade zeta function  $\zeta_P$ . Main results on this function will be

proved in Section 8. The Poissonian case includes all phases in which the first level is in the glassy (G) phase, the set  $E_1 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$ , and the boundaries between these sets. Section 9 contains some preliminary results on the first level of the GREM. Results on the fluctuations of  $\mathcal{Z}_n(\beta)$  in phases of the form  $G^{d_1} E^{d-d_1}$  with  $1 \leq d_1 \leq d$ , as well as in the set  $E_1 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$ , will be proved in Section 11 after an essential part of the work has been done in Section 10. Section 12 deals with boundaries separating phases of the form  $G^{d_1} E^{d-d_1}$ . Results on the fluctuations of  $\mathcal{Z}_n(\beta)$  in phases involving at least one level in fluctuation (F) phase will be proved in Section 13.

Let us finally make a remark on the contributions of the levels  $2, \dots, d$ . Since  $\tilde{\mathcal{Z}}_{n,k}(\beta)$  has the same structure as  $\mathcal{Z}_n(\beta)$ , it is natural to use induction over  $d$ , the number of levels of the GREM. Then, the induction assumption provides information about  $\tilde{\mathcal{Z}}_{n,k}(\beta)$ . For technical reasons, we will frequently need to obtain estimates on the moments of  $\tilde{\mathcal{Z}}_{n,k}(\beta)$ . To prove such estimates, we also use induction. It is useful to keep in mind the following principle: the moment properties of  $\tilde{\mathcal{Z}}_{n,k}(\beta)$  are usually better than those of  $\mathcal{Z}_n(\beta)$ . The reason for this is the standing assumption (1.9).

Only after the fluctuations of  $\mathcal{Z}_n(\beta)$  at every complex  $\beta$  have been identified, we will be able to prove Theorem 2.1 (on the limiting log-partition function) and Theorem 2.3 (on the global distribution of complex zeros). One may ask whether it is possible to prove Theorem 2.1 directly, without computing the fluctuations of  $\mathcal{Z}_n(\beta)$ . As we explained in Section 2.1, it is not difficult to *guess* the formula for the limiting log-partition function. However, we do not know any rigorous *proof* of this formula which avoids the computation of the fluctuations of  $\mathcal{Z}_n(\beta)$ . The main difficulty is that in order to obtain a lower estimate on  $|\mathcal{Z}_n(\beta)|$  we need to control the possible cancellations among the terms in the definition of  $\mathcal{Z}_n(\beta)$ , a problem which does not appear in the real temperature case. Our way to control the cancellations is to find the limiting distribution and to show that it has no atom at zero.

### 3. AUXILIARY RESULTS

In this section, we collect a number of mostly well-known results which will be frequently used in the sequel. The reader may skip this section and return to it later if necessary.

**3.1. Inequalities for the moments of random variables.** In our proofs, we will often need estimates for the moments of random variables. In this section, we collect such estimates. For example, we will frequently use Lyapunov's inequality: For every real or complex random variable  $X$  and every  $0 < s \leq t$  it holds that

$$(3.1) \quad (\mathbb{E}|X|^s)^{1/s} \leq (\mathbb{E}|X|^t)^{1/t}.$$

For arbitrary (deterministic) numbers  $x_1, \dots, x_m \in \mathbb{C}$  and  $p > 0$ , it holds that

$$(3.2) \quad |x_1 + \dots + x_m|^p \leq \max(1, m^{p-1}) \sum_{i=1}^m |x_i|^p.$$

In the case  $p \geq 1$ , this follows from Jensen's inequality, whereas in the case  $0 < p \leq 1$  the inequality is easy to prove by induction.

**Lemma 3.1.** *For  $p \geq 0$ , and any complex-valued random variable  $Z$ ,*

$$(3.3) \quad \mathbb{E}|Z - \mathbb{E}Z|^p \leq \max(1, 2^{p-1})(\mathbb{E}|Z|^p + |\mathbb{E}Z|^p).$$

*For  $p \geq 1$  we even have  $\mathbb{E}|Z - \mathbb{E}Z|^p \leq 2^p \mathbb{E}|Z|^p$ .*

*Proof.* Inequality (3.3) follows from (3.2). For  $p \geq 1$  we have  $|\mathbb{E}Z|^p \leq (\mathbb{E}|Z|)^p \leq \mathbb{E}|Z|^p$  by (3.1). This proves the second statement of the lemma.  $\square$

The next proposition (see [47] or [35, Chapter 2.3]) is an immediate corollary of (3.2).

**Proposition 3.2.** *Let  $\eta_1, \dots, \eta_n$  be arbitrary (not necessarily independent) real or complex random variables with finite  $p$ -th absolute moment, where  $0 < p \leq 1$ . Then,*

$$(3.4) \quad \mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq \sum_{k=1}^n \mathbb{E}|\eta_k|^p.$$

Von Bahr and Esseen [47] showed that up to a multiplicative constant, inequality (3.4) remains true for  $1 \leq p \leq 2$ , if we additionally assume that the variables are independent and centered.

**Proposition 3.3** (von Bahr–Esseen inequality [47]). *Let  $\eta_1, \dots, \eta_n$  be centered, independent real or complex random variables with finite  $p$ -th absolute moment, where  $1 \leq p \leq 2$ . Then,*

$$(3.5) \quad \mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq 2 \sum_{k=1}^n \mathbb{E}|\eta_k|^p.$$

We need a similar estimate for not necessarily centered random variables.

**Proposition 3.4.** *Let  $\eta_1, \dots, \eta_n$  be independent real or complex random variables with finite  $p$ -th absolute moment, where  $1 \leq p \leq 2$ . Let also  $m_k = \mathbb{E}\eta_k$  and  $m = m_1 + \dots + m_n$ . Then,*

$$\mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq 2^{2p-1} \sum_{k=1}^n \mathbb{E}|\eta_k|^p + 2^{p-1}|m|^p.$$

*Proof.* The random variables  $\tilde{\eta}_k := \eta_k - m_k$  are centered. Using Jensen's inequality (3.2) and applying the von Bahr–Esseen inequality to  $\tilde{\eta}_k$ , we obtain

$$\mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq 2^{p-1} \mathbb{E}|\tilde{\eta}_1 + \dots + \tilde{\eta}_n|^p + 2^{p-1}|m|^p \leq 2^p \sum_{k=1}^n \mathbb{E}|\tilde{\eta}_k|^p + 2^{p-1}|m|^p.$$

The proof is completed by noting that  $\mathbb{E}|\tilde{\eta}_k|^p \leq 2^p \mathbb{E}|\eta_k|^p$  by Lemma 3.1.  $\square$

For  $p = 2$ , the von Bahr–Esseen inequality is trivially satisfied by the additivity property of the variance, however, for  $p > 2$ , it is, in general, not valid. Instead, we have the following result.

**Proposition 3.5** (Rosenthal inequality [38]). *Let  $\eta_1, \dots, \eta_n$  be centered, independent real or complex random variables with finite  $p$ -th absolute moment, where  $p \geq 2$ . Then,*

$$\mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq K_p \max \left\{ \sum_{k=1}^n \mathbb{E}|\eta_k|^p, \sum_{k=1}^n (\mathbb{E}|\eta_k|^2)^{p/2} \right\},$$

where  $K_p$  is a constant depending only on  $p$  (and not depending on  $n$  or on the distribution of the  $\eta_k$ 's).

We need a version of this inequality which is valid for not necessarily centered random variables.

**Proposition 3.6.** *Let  $\eta_1, \dots, \eta_n$  be independent real or complex random variables with finite  $p$ -th absolute moment, where  $p \geq 2$ . Let also  $m_k = \mathbb{E}\eta_k$  and  $m = m_1 + \dots + m_n$ . Then,*

$$\mathbb{E}|\eta_1 + \dots + \eta_n|^p \leq K'_p \max \left\{ \sum_{k=1}^n \mathbb{E}|\eta_k|^p, \left( \sum_{k=1}^n \mathbb{E}|\eta_k|^2 \right)^{p/2} \right\} + K'_p |m|^p,$$

where  $K'_p$  is a constant depending only on  $p$  (and not depending on  $n$  or on the distribution of the  $\eta_k$ 's).

*Proof.* The random variables  $\tilde{\eta}_k := \eta_k - m_k$  are centered. Using Jensen's inequality (3.2) and applying the Rosenthal inequality to  $\tilde{\eta}_k$ , we obtain

$$\begin{aligned} \mathbb{E}|\eta_1 + \dots + \eta_n|^p &\leq 2^{p-1} \mathbb{E}|\tilde{\eta}_1 + \dots + \tilde{\eta}_n|^p + 2^{p-1} |m|^p \\ &\leq 2^{p-1} K_p \max \left\{ \sum_{k=1}^n \mathbb{E}|\tilde{\eta}_k|^p, \left( \sum_{k=1}^n \mathbb{E}|\tilde{\eta}_k|^2 \right)^{p/2} \right\} + 2^{p-1} |m|^p. \end{aligned}$$

To complete the proof, note that  $\mathbb{E}|\tilde{\eta}_k|^p \leq 2^p \mathbb{E}|\eta_k|^p$  by Lemma 3.1 and that  $\mathbb{E}|\tilde{\eta}_k|^2 = \mathbb{E}|\eta_k|^2 - |m_k|^2 \leq \mathbb{E}|\eta_k|^2$ .  $\square$

**3.2. Truncated exponential moments of the normal distribution.** In this section we recall several well-known properties of the Gaussian distribution. Let  $\xi \sim N_{\mathbb{R}}(0, 1)$  be a real standard normal random variable. Let

$$\varphi(t) = e^{-\frac{1}{2}t^2}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt, \quad \bar{\Phi}(x) = 1 - \Phi(x)$$

be the density, the distribution function, and the tail function of  $\xi$ , respectively.

**Lemma 3.7** (Mills ratio inequality). *For every  $x > 0$ ,  $\bar{\Phi}(x) \leq \frac{1}{x} \varphi(x)$ .*

*Proof.* Using the definition of  $\bar{\Phi}$  and introducing the variable  $s := t - x$  we obtain

$$\bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt = \varphi(x) \int_0^\infty e^{-\frac{s^2}{2} - xs} ds \leq \varphi(x) \int_0^\infty e^{-xs} ds = \frac{\varphi(x)}{x}.$$

This is the desired inequality.  $\square$

The next lemma follows from a simple change of variables; see [25, Lemma 3.3].

**Lemma 3.8.** *Let  $\xi \sim N_{\mathbb{R}}(0, 1)$ . For every  $a \in \mathbb{R}$ ,  $w \in \mathbb{C}$ ,*

- (1)  $\mathbb{E}[e^{w\xi} \mathbb{1}_{\xi > a}] = e^{\frac{w^2}{2}} \bar{\Phi}(a - w).$
- (2)  $\mathbb{E}[e^{w\xi} \mathbb{1}_{\xi < a}] = e^{\frac{w^2}{2}} \Phi(a - w).$

**Lemma 3.9.** *Let  $\xi \sim N_{\mathbb{R}}(0, 1)$  and  $a, w \in \mathbb{R}$ .*

- (1) *For  $a > w$ ,  $\mathbb{E}[e^{w(\xi-a)} \mathbb{1}_{\xi > a}] \leq \frac{1}{\sqrt{2\pi(a-w)}} e^{-a^2/2}.$*
- (2) *For  $a < w$ ,  $\mathbb{E}[e^{w(\xi-a)} \mathbb{1}_{\xi < a}] \leq \frac{1}{\sqrt{2\pi(w-a)}} e^{-a^2/2}.$*

*Proof of (1).* By Lemma 3.8, we have  $\mathbb{E}[e^{w(\xi-a)} \mathbb{1}_{\xi > a}] = e^{\frac{w^2}{2} - aw} \bar{\Phi}(a-w)$ . Applying Lemma 3.7 to the right-hand side, we obtain the desired inequality. The proof of (2) is analogous.  $\square$

The function  $\Phi$  admits an analytic continuation to the entire complex plane. The following lemma gives the well-known complex plane asymptotics of  $\Phi$ . It is standard, see, e.g., [1, Eq. 7.1.23 on p. 298] for (3.6) and [36, Chapter IV, Problem 189] for (3.7). We will sketch the proof, since the lemma will be crucial when establishing the beak shaped form of the phases  $G^{d_1} E^{d_3}$ .

**Lemma 3.10.** *Fix some  $\varepsilon > 0$ . The following asymptotics hold uniformly in the region specified below as  $|z| \rightarrow \infty$ :*

$$(3.6) \quad \Phi(z) = \begin{cases} -\frac{1+o(1)}{\sqrt{2\pi z}} e^{-\frac{z^2}{2}}, & \text{if } |\arg z| > \frac{\pi}{4} + \varepsilon, \\ 1 - \frac{1+o(1)}{\sqrt{2\pi z}} e^{-\frac{z^2}{2}}, & \text{if } |\arg z| < \frac{3\pi}{4} - \varepsilon. \end{cases}$$

In particular,

$$(3.7) \quad \Phi(z) \rightarrow \begin{cases} 1, & \text{if } |\arg z| < \frac{\pi}{4} - \varepsilon, \\ 0, & \text{if } |\arg z| > \frac{3\pi}{4} + \varepsilon, \\ \infty, & \text{if } \frac{\pi}{4} + \varepsilon < |\arg z| < \frac{3\pi}{4} - \varepsilon. \end{cases}$$

**Remark 3.11.** We take the principal value of the argument, ranging in  $(-\pi, \pi]$  and having a jump discontinuity on the negative half-axis. In the domain  $\frac{\pi}{4} + \varepsilon < |\arg z| < \frac{3\pi}{4} - \varepsilon$  both asymptotics in (3.6) can be applied and give the same result.

*Proof of Lemma 3.10.* We prove the second case of (3.6). The analytic continuation of the function  $\bar{\Phi}(z) = 1 - \Phi(z)$  is given by

$$\bar{\Phi}(z) = \frac{1}{\sqrt{2\pi}} \int_{\gamma_z} e^{-\frac{s^2}{2}} ds,$$

where, for the time being,  $\gamma_z$  is the horizontal ray connecting  $z$  to  $i \operatorname{Im} z + \infty$ . However, since the function  $e^{-s^2/2}$  converges to 0 exponentially fast for  $|\arg s| < \frac{\pi}{4}$ , we can rotate  $\gamma_z$  by any angle  $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$  without changing the integral. Let us agree to choose  $\theta$  in the following way:

$$\theta = \begin{cases} 0, & \text{if } |\arg z| < \frac{\pi}{2} - \varepsilon, \\ -\frac{\pi}{4} + \frac{\varepsilon}{2}, & \text{if } \arg z \in (\frac{\pi}{4} + \varepsilon, \frac{3\pi}{4} - \varepsilon) \\ \frac{\pi}{4} - \frac{\varepsilon}{2}, & \text{if } \arg z \in (-\frac{3\pi}{4} + \varepsilon, -\frac{\pi}{4} - \varepsilon). \end{cases}$$

Note that the domain  $|\arg z| < \frac{3\pi}{4} - \varepsilon$  is completely covered (with overlaps) by these 3 cases. We parametrize the contour  $\gamma_z$  as follows:

$$s = \gamma_z(t) = z + e^{i\theta} t / |z|, \quad t \geq 0.$$

Then, the integral for  $\bar{\Phi}(z)$  takes the form

$$\bar{\Phi}(z) = \left( \frac{1}{\sqrt{2\pi z}} e^{-\frac{z^2}{2}} \right) \int_0^\infty \omega(z) e^{-\omega(z)t - \frac{1}{2} e^{2i\theta} \frac{t^2}{|z|^2}} dt =: \left( \frac{1}{\sqrt{2\pi z}} e^{-\frac{z^2}{2}} \right) I(z),$$

where  $\omega(z) = e^{i\theta} z / |z|$ . The above choice of  $\theta$  guarantees that  $|\arg \omega(z)| < \frac{\pi}{2} - \frac{\varepsilon}{2}$ . It is an elementary exercise to show that under this restriction, the integral  $I(z)$  converges uniformly to 1 as  $|z| \rightarrow \infty$ . This proves the second case of (3.6).

The first case of (3.6) follows from the second case and the formula  $\Phi(z) = \bar{\Phi}(-z)$ . To prove (3.7), use (3.6) and note that  $e^{-z^2/2}$  uniformly converges to 0 as  $|z| \rightarrow \infty$  in such a way that  $|\arg z| < \frac{\pi}{4} - \varepsilon$  or  $|\arg z| > \frac{3\pi}{4} + \varepsilon$ .  $\square$

**3.3. Results on weak convergence.** Let  $D \subset \mathbb{C}$  be a connected open set. A family of random continuous or analytic functions on  $D$  is called tight if every sequence from this family contains a weakly convergent subsequence. Criteria for tightness in the space  $C(D)$  are well-known; see [2, Theorems 8.2, 8.3]. These criteria simplify considerably if we are dealing with *analytic* (rather than merely continuous) functions. The next lemma can be found in [43, the remark after Lemma 2.6]; see also [25, Lemma 4.2] for a slightly weaker result.

**Proposition 3.12.** *Let  $Z_1, Z_2, \dots$  be a sequence of random analytic functions on a connected open set  $D \subset \mathbb{C}$ . Assume that there is a  $p > 0$  and a locally integrable function  $F: D \rightarrow \mathbb{R}$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}|Z_n(\beta)|^p < F(\beta) \text{ for all } \beta \in D.$$

*Then, the sequence  $Z_1, Z_2, \dots$  is tight on  $\mathcal{H}(D)$ .*

The next proposition, see [25, Lemma 4.3] or [43, Proposition 2.3], states that the weak convergence of random analytic functions implies the weak convergence of the corresponding point processes of zeros. It is essential that the functions are analytic, not merely continuous.

**Proposition 3.13.** *Let  $Z_1, Z_2, \dots$  be a sequence of random analytic functions on  $D$  converging to some random analytic function  $Z$  weakly on  $\mathcal{H}(D)$ . Assume that  $Z$  is not identically zero, with probability 1. Then, the following convergence of point processes holds weakly on  $\mathcal{N}(D)$ :*

$$\mathbf{Zeros}\{Z_n(\beta): \beta \in D\} \xrightarrow[n \rightarrow \infty]{w} \mathbf{Zeros}\{Z(\beta): \beta \in D\}.$$

For the next proposition, we refer to [26, Proposition 14.6].

**Proposition 3.14.** *A sequence of random continuous functions  $Z_1, Z_2, \dots$  converges weakly to some random continuous function  $Z$  on  $C(D)$  if and only if for every compact set  $K \subset D$  the restriction of  $Z_n$  to  $K$  converges to the restriction of  $Z$  to  $K$  weakly on  $C(K)$ .*

The next lemma is standard; see Theorem 4.2 on p. 25 in [2].

**Lemma 3.15.** *For every  $n \in \mathbb{N} \cup \{\infty\}$ , let  $\mathbf{S}_n$  and  $\mathbf{S}_{n,T}$ ,  $T \in \mathbb{N}$ , be random elements defined on a probability space  $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$  and taking values in a separable metric space  $(A, \rho)$ . Assume that*

- (1) *For every  $T \in \mathbb{N}$ ,  $\mathbf{S}_{n,T}$  converges weakly to  $\mathbf{S}_{\infty,T}$ , as  $n \rightarrow \infty$ .*
- (2) *For every  $\varepsilon > 0$ , we have  $\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}_n[\rho(\mathbf{S}_n, \mathbf{S}_{n,T}) > \varepsilon] = 0$ .*
- (3) *For every  $\varepsilon > 0$ , we have  $\lim_{T \rightarrow \infty} \mathbb{P}_\infty[\rho(\mathbf{S}_\infty, \mathbf{S}_{\infty,T}) > \varepsilon] = 0$ .*

*Then,  $\mathbf{S}_n$  converges weakly to  $\mathbf{S}_\infty$ , as  $n \rightarrow \infty$ .*



**Remark 3.16.** Lemma 3.15 is illustrated by the following diagram:

$$\begin{array}{ccc}
 \mathbf{S}_{n,T} & \xrightarrow[n \rightarrow \infty]{w} & \mathbf{S}_{\infty,T} \\
 \downarrow P & & \downarrow P \\
 \mathbf{S}_n & \xrightarrow[n \rightarrow \infty]{w} & \mathbf{S}_{\infty}
 \end{array}$$

(Note: The vertical arrows from  $\mathbf{S}_{n,T}$  to  $\mathbf{S}_n$  and from  $\mathbf{S}_{\infty,T}$  to  $\mathbf{S}_{\infty}$  are labeled  $T, n \downarrow \infty$  and  $T \downarrow \infty$  respectively.)

We will apply Lemma 3.15 many times in the proofs of functional limit theorems. In our context,  $\mathbf{S}_n$  will be a normalized version of the partition function  $Z_n$ , whereas  $\mathbf{S}_{n,T}$  will be a truncated version of  $\mathbf{S}_n$ , with  $T$  being a truncation parameter.

The next lemma will be used in the proof of Theorem 2.30.

**Lemma 3.17.** *Let  $Z_1, Z_2 \dots$  be a sequence of random continuous functions on  $\mathbb{C}$  converging weakly to a random continuous function  $Z$  on  $\mathbb{C}$ . Let  $\beta_* \in \mathbb{C}$  be fixed and let  $\beta_n \in \mathbb{C}$  and  $q_n \in \mathbb{C}$  be sequences such that  $\lim_{n \rightarrow \infty} \beta_n = \beta_*$  and  $\lim_{n \rightarrow \infty} q_n = 0$ . Then, weakly on  $C(\mathbb{C})$  it holds that*

$$\{Z_n(\beta_n + q_n t) : t \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \{Z(\beta_*) : t \in \mathbb{C}\}.$$

*Proof.* Define the mappings  $\psi_n, \psi : C(\mathbb{C}) \rightarrow C(\mathbb{C})$  by

$$\psi_n(f)(t) = f(\beta_n + q_n t), \quad \psi(f)(t) = f(\beta_*) \quad \text{for } f \in C(\mathbb{C}), \quad t \in \mathbb{C}.$$

If  $f_n \in C(\mathbb{C})$  is a sequence converging to  $f \in C(\mathbb{C})$  locally uniformly, then it is easy to check that  $\psi_n(f_n)$  converges to  $\psi(f)$  locally uniformly. By the continuous mapping theorem, see Theorem 3.27 in [26], we obtain that  $\psi_n(Z_n)$  converges to  $\psi(Z)$  weakly on  $C(\mathbb{C})$ . This proves the lemma.  $\square$

**3.4. Central limit theorems for triangular arrays of random vectors.** We will often use classical results on limiting distributions for sums of independent random vectors. Specifically, to prove central limit theorems in the case  $|\sigma| < \frac{\sigma_1}{2}$  we will need Lyapunov's central limit theorem. Let  $k_n \in \mathbb{N}$  be a sequence such that  $\lim_{n \rightarrow \infty} k_n = \infty$ .

**Theorem 3.18.** *For every  $n \in \mathbb{N}$ , let  $\{\mathbf{Z}_{n,k} : 1 \leq k \leq k_n\}$  be independent  $\mathbb{R}^m$ -valued random vectors written as  $\mathbf{Z}_{n,k} = \{\mathbf{Z}_{n,k}(i)\}_{i=1}^m$ . Let  $\mathbf{S}_n^* = \sum_{k=1}^{k_n} (\mathbf{Z}_{n,k} - \mathbb{E}\mathbf{Z}_{n,k})$ . Assume that*

- (1) *The covariance matrix of  $\mathbf{S}_n^*$  converges as  $n \rightarrow \infty$  to some matrix  $\Sigma$ .*
- (2) *The Lyapunov condition is satisfied: For some  $\delta > 0$ ,*

$$(3.8) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \mathbb{E}|\mathbf{Z}_{n,k}(i)|^{2+\delta} = 0 \quad \text{for all } 1 \leq i \leq m.$$

*Then,  $\mathbf{S}_n^*$  converges in distribution to a mean zero Gaussian distribution on  $\mathbb{R}^m$  with mean 0 and covariance matrix  $\Sigma$ .*

**Remark 3.19.** Usually, the Lyapunov condition is stated in the form

$$(3.9) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \mathbb{E}|\mathbf{Z}_{n,k} - \mathbb{E}\mathbf{Z}_{n,k}|^{2+\delta} = 0 \quad \text{for all } 1 \leq i \leq m.$$

However, it is easy to see using Lemma 3.1 that (3.8) implies (3.9).

The following result, see [32, Theorem 3.2.2], is somewhat more general than the Lyapunov (and even Lindeberg) central limit theorem. We will need it in the case  $|\sigma| = \frac{\sigma_1}{2}$ . We denote by  $|\cdot|$  the Euclidean norm in  $\mathbb{R}^m$  and by  $\text{Cov } Z$  the covariance matrix of an  $\mathbb{R}^m$ -valued random vector  $Z$ .

**Theorem 3.20.** *For every  $n \in \mathbb{N}$ , let  $\{\mathbf{Z}_{n,k} : 1 \leq k \leq k_n\}$  be independent  $\mathbb{R}^m$ -valued random vectors. Assume that the following conditions hold:*

- (1) *For every  $\varepsilon > 0$ ,  $\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \mathbb{P}[|\mathbf{Z}_{n,k}| > \varepsilon] = 0$ .*
- (2) *For some positive semidefinite matrix  $\Sigma$ ,*

$$\Sigma = \lim_{\varepsilon \downarrow 0} \limsup_{n \rightarrow \infty} \sum_{k=1}^{k_n} \text{Cov}[\mathbf{Z}_{n,k} \mathbb{1}_{|\mathbf{Z}_{n,k}| < \varepsilon}] = \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} \sum_{k=1}^{k_n} \text{Cov}[\mathbf{Z}_{n,k} \mathbb{1}_{|\mathbf{Z}_{n,k}| < \varepsilon}].$$

*Then, the random vector  $\mathbf{S}_n := \sum_{k=1}^{k_n} (\mathbf{Z}_{n,k} - \mathbb{E}[\mathbf{Z}_{n,k} \mathbb{1}_{|\mathbf{Z}_{n,k}| < R}])$  converges weakly to a mean zero Gaussian distribution on  $\mathbb{R}^m$  with covariance matrix  $\Sigma$ . Here,  $R > 0$  is arbitrary.*

#### 4. PROOF OF THE CENTRAL LIMIT THEOREM IN THE STRIP $|\sigma| < \frac{\sigma_1}{2}$

**4.1. Proof of Theorem 2.4.** Let  $\beta = \sigma + i\tau \in \mathbb{C} \setminus \{0\}$  be such that  $|\sigma| < \frac{\sigma_1}{2}$ . For a complex-valued random variable  $Z$ , the variance is defined by  $\text{Var } Z = \mathbb{E}|Z|^2 - |\mathbb{E}Z|^2$ . Our aim is to prove Theorem 2.4 which states that

$$(4.1) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \begin{cases} N_{\mathbb{C}}(0, 1), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0, 1), & \text{if } \tau = 0. \end{cases}$$

The idea of the proof is to split  $\mathcal{Z}_n(\beta)$  into the sum of the contributions of the first level multiplied by the contributions of all other levels. We can write

$$\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta) = \sum_{k=1}^{N_{n,1}} W_{n,k}^*,$$

where for every  $n \in \mathbb{N}$ ,  $\{W_{n,k}^* : 1 \leq k \leq N_{n,1}\}$  are i.i.d. random variables defined by

$$(4.2) \quad W_{n,k}^* = X_{n,k} Y_{n,k} - \mathbb{E}[X_{n,k} Y_{n,k}].$$

Here,  $X_{n,k}$  (the contributions of the first level) and  $Y_{n,k}$  (the contributions of the remaining levels) are given by

$$(4.3) \quad X_{n,k} = e^{\beta \sqrt{na_1} \xi_k}, \quad Y_{n,k} = \sum_{\varepsilon_2=1}^{N_{n,2}} \dots \sum_{\varepsilon_d=1}^{N_{n,d}} e^{\beta \sqrt{n}(\sqrt{a_2} \xi_{k\varepsilon_2} + \dots + \sqrt{a_d} \xi_{k\varepsilon_2 \dots \varepsilon_d})}.$$

Note that for every  $k$ , the random variable  $Y_{n,k}$  has the same structure as  $\mathcal{Z}_n(\beta)$  but with  $d-1$  instead of  $d$  levels. Also, note that both families  $\{X_{n,k} : 1 \leq k \leq N_{n,1}\}$  and  $\{Y_{n,k} : 1 \leq k \leq N_{n,1}\}$  consist of i.i.d. random variables, and that there is no dependence between these families.

Let  $z_n^2 = \text{Var } \mathcal{Z}_n(\beta)$ . Our aim is to show that the random variable  $\sum_{k=1}^{N_{n,1}} z_n^{-1} W_{n,k}^*$  converges in distribution to a standard normal random variable (which may be real or complex depending on whether  $\tau = 0$  or  $\tau \neq 0$ ). We will show that the triangular array

$$\{z_n^{-1} W_{n,k}^* : 1 \leq k \leq N_{n,1}, n \in \mathbb{N}\}$$

satisfies the conditions of the Lyapunov central limit theorem; see Theorem 3.18. We view each  $W_{n,k}^*$  as a two-dimensional random vector  $(\text{Re } W_{n,k}^*, \text{Im } W_{n,k}^*)$ . To

simplify the notation, let  $(W_n^*, X_n, Y_n)$  be random variables having the same (joint) law as any of the  $(W_{n,k}^*, X_{n,k}, Y_{n,k})$ . Note that  $\mathbb{E}W_n^* = 0$ , by (4.2).

STEP 1. In the first step, we will compute the asymptotics of the covariance matrix of the vector  $W_n^*$  given by

$$\text{Cov } W_n^* = \begin{pmatrix} \mathbb{E}[(\text{Re } W_n^*)^2] & \mathbb{E}[\text{Re } W_n^* \text{Im } W_n^*] \\ \mathbb{E}[\text{Re } W_n^* \text{Im } W_n^*] & \mathbb{E}[(\text{Im } W_n^*)^2] \end{pmatrix}.$$

Namely, we will show that

$$(4.4) \quad \lim_{n \rightarrow \infty} N_{n,1} z_n^{-2} \text{Cov } W_n^* = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{if } \tau \neq 0,$$

$$(4.5) \quad \lim_{n \rightarrow \infty} N_{n,1} z_n^{-2} \text{Cov } W_n^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{if } \tau = 0.$$

**Lemma 4.1.** *For  $\tau \neq 0$ ,  $\mathbb{E}[W_n^{*2}] = o(\mathbb{E}|W_n^*|^2)$ .*

*Proof.* We have, due to the independence of  $X_n$  and  $Y_n$ ,

$$(4.6) \quad \mathbb{E}[W_n^{*2}] = \mathbb{E}X_n^2 \mathbb{E}Y_n^2 - (\mathbb{E}X_n \mathbb{E}Y_n)^2,$$

$$(4.7) \quad \mathbb{E}|W_n^*|^2 = \mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2 - |\mathbb{E}X_n|^2 |\mathbb{E}Y_n|^2.$$

We will show that the term  $\mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2$  asymptotically dominates all other terms in these equalities. We have

$$\mathbb{E}X_n^2 = \mathbb{E}e^{2\beta\sqrt{na_1}\xi} = e^{2\beta^2 a_1 n}, \quad \mathbb{E}|X_n|^2 = \mathbb{E}e^{2\sigma\sqrt{na_1}\xi} = e^{2\sigma^2 a_1 n}.$$

Since  $\text{Re}(\beta^2) = \sigma^2 - \tau^2 < \sigma^2$  for  $\tau \neq 0$ , we have  $\mathbb{E}X_n^2 = o(\mathbb{E}|X_n|^2)$ . Similarly,

$$(4.8) \quad |\mathbb{E}X_n|^2 = |e^{\frac{1}{2}\beta^2 a_1 n}|^2 = e^{(\sigma^2 - \tau^2) a_1 n} = o(\mathbb{E}|X_n|^2) \text{ for } \beta \neq 0.$$

Also, we have the inequalities  $|\mathbb{E}Y_n^2| \leq \mathbb{E}|Y_n|^2$  and  $|(\mathbb{E}Y_n)^2| = |\mathbb{E}Y_n|^2 \leq \mathbb{E}|Y_n|^2$ . Inserting all these results into (4.6) and (4.7) yields  $\mathbb{E}[W_n^{*2}] = o(\mathbb{E}|W_n^*|^2)$ .  $\square$

Recall that  $z_n^2 = \mathbb{E}|\sum_{k=1}^{N_{n,1}} W_{n,k}^*|^2$  and  $\mathbb{E}W_{n,k}^* = 0$ . Hence,

$$(4.9) \quad z_n^2 = N_{n,1} \mathbb{E}|W_n^*|^2 = N_{n,1} (\mathbb{E}(\text{Re } W_n^*)^2 + \mathbb{E}(\text{Im } W_n^*)^2).$$

In the case  $\tau = 0$ , we have  $\text{Im } W_n^* = 0$  which immediately yields (4.5). In the case  $\tau \neq 0$ , we have by Lemma 4.1,

$$(4.10) \quad \mathbb{E}(\text{Re } W_n^*)^2 - \mathbb{E}(\text{Im } W_n^*)^2 = \text{Re } \mathbb{E}[W_n^{*2}] = o(\mathbb{E}|W_n^*|^2),$$

$$(4.11) \quad 2 \mathbb{E}(\text{Re } W_n^* \text{Im } W_n^*) = \text{Im } \mathbb{E}[W_n^{*2}] = o(\mathbb{E}|W_n^*|^2),$$

From (4.9), (4.10), (4.11), we obtain that (4.4) holds in the case  $\tau \neq 0$ .

As a byproduct of Lemma 4.1, see (4.7) and (4.8), we proved the following

**Lemma 4.2.** *For  $\beta \in \mathbb{C} \setminus \{0\}$ ,  $\mathbb{E}|W_n^*|^2 \sim \mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2$  and*

$$z_n^2 = \text{Var } \mathcal{Z}_n(\beta) \sim N_{n,1} \mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2.$$

STEP 2. We will now verify the Lyapunov condition. Choose some  $2 < p < \frac{\log \alpha_1}{a_1 \sigma^2}$ . This is possible by the assumption  $2|\sigma| < \sigma_1$ . We will verify that

$$(4.12) \quad N_{n,1} \mathbb{E}|W_n^*|^p = o(z_n^p).$$

In view of the inequality  $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ , it suffices to verify that

$$(L1) \quad N_{n,1} \mathbb{E}|X_n|^p \mathbb{E}|Y_n|^p = o(N_{n,1}^{p/2} (\mathbb{E}|X_n|^2)^{p/2} (\mathbb{E}|Y_n|^2)^{p/2}).$$

$$(L2) \quad \alpha_1^n |\mathbb{E}X_n|^p |\mathbb{E}Y_n|^p = o(N_{n,1}^{p/2} (\mathbb{E}|X_n|^2)^{p/2} (\mathbb{E}|Y_n|^2)^{p/2}).$$

Since (L1) implies (L2) by the Jensen inequality, we need to verify (L1) only. This will be done in Lemmas 4.3 and 4.4, below.

**Lemma 4.3.** *Let  $2 < p < \frac{\log \alpha_1}{a_1 \sigma^2}$ . Then,  $N_{n,1} \mathbb{E}|X_n|^p = o(N_{n,1}^{p/2} (\mathbb{E}|X_n|^2)^{p/2})$ .*

*Proof.* We have  $\mathbb{E}|X_n|^p = e^{\frac{1}{2}\sigma^2 p^2 a_1 n}$  and the lemma follows immediately from the inequality

$$\log \alpha_1 + \frac{1}{2}\sigma^2 p^2 a_1 < \frac{1}{2}p \log \alpha_1 + \sigma^2 p a_1$$

which, in turn, is a consequence of the assumption  $2 < p < \frac{\log \alpha_1}{a_1 \sigma^2}$ .  $\square$

**Lemma 4.4.** *Let  $2 < p < \frac{\log \alpha_2}{a_2 \sigma^2}$ . Then,  $\mathbb{E}|Y_n|^p = O((\mathbb{E}|Y_n|^2)^{p/2})$ .*

*Proof.* The proof is by induction over  $d$ . For  $d = 1$  we have  $Y_n = 1$  and the statement is true. Suppose that the inequality is true in the setting of  $d$  levels. We need to verify it for  $d + 1$  levels. However, the analogue of  $Y_n$  for  $d + 1$  levels is  $\mathcal{Z}_n(\beta)$ . That is, we need to show that

$$(4.13) \quad \mathbb{E}|\mathcal{Z}_n(\beta)|^p < C(\mathbb{E}|\mathcal{Z}_n(\beta)|^2)^{p/2} \quad \text{for } 2 < p < \frac{\log \alpha_1}{a_1 \sigma^2}.$$

To this end, we apply Proposition 3.6 to the random variables  $\eta_k = X_{k,n} Y_{k,n}$ :

$$\mathbb{E}|\mathcal{Z}_n(\beta)|^p \leq K'_p \max\{N_{n,1} \mathbb{E}|X_n|^p \mathbb{E}|Y_n|^p, (N_{n,1} \mathbb{E}|X_n Y_n|^2)^{p/2}\} + |\mathbb{E}\mathcal{Z}_n(\beta)|^p.$$

To complete the proof of (4.13), we need to show that

- (A1)  $N_{n,1} \mathbb{E}|X_n|^p \mathbb{E}|Y_n|^p \leq C(\mathbb{E}|\mathcal{Z}_n(\beta)|^2)^{p/2}$ .
- (A2)  $N_{n,1} \mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2 \leq C \mathbb{E}|\mathcal{Z}_n(\beta)|^2$ .
- (A3)  $|\mathbb{E}\mathcal{Z}_n(\beta)|^p \leq C(\mathbb{E}|\mathcal{Z}_n(\beta)|^2)^{p/2}$ .

Condition (A1) follows from the induction assumption together with Lemma 4.3. Condition (A3) is an immediate consequence of Jensen's inequality (since  $p > 2$ ). The left-hand side of (A2) is asymptotic to  $\text{Var } \mathcal{Z}_n(\beta)$  by Lemma 4.2, which proves Condition (A2).  $\square$

## 5. PROOF OF THE CENTRAL LIMIT THEOREM FOR $|\sigma| = \frac{\sigma_1}{2}$

**5.1. Proof of Theorem 2.10.** Let  $\beta = \beta(n) = \sigma(n) + i\tau$  be such that  $\tau \in \mathbb{R}$  is constant and  $\sigma = \sigma(n)$  depends on  $n$ . Assume that for some  $u \in \mathbb{R}$ ,

$$(5.1) \quad \sigma(n) = \frac{\sigma_1}{2} - \frac{u}{2\sqrt{na_1}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Our aim is to prove Theorem 2.10 which states that

$$(5.2) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \begin{cases} N_{\mathbb{C}}(0, \Phi(u)), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0, \Phi(u)), & \text{if } \tau = 0. \end{cases}$$

**STEP 0.** We start by introducing some notation. As in Section 4, we represent  $\mathcal{Z}_n(\beta)$  as a sum of the contributions of the first level multiplied by the contributions of all other levels:

$$\mathcal{Z}_n(\beta) = \sum_{k=1}^{N_{n,1}} W_{n,k}, \quad W_{n,k} = X_{n,k} Y_{n,k},$$

where  $X_{n,k} = e^{\beta\sqrt{na_1}\xi_k}$  (the contributions of the first level) and  $Y_{n,k}$  (the contributions of the remaining levels) are defined in the same way as in (4.3). Define

$$(5.3) \quad X'_{n,k} = \frac{X_{n,k}}{\sqrt{\mathbb{E}|X_{n,k}|^2}} = e^{\beta\sqrt{na_1}\xi_k - \sigma^2 na_1}, \quad Y'_{n,k} = \frac{Y_{n,k}}{\sqrt{\mathbb{E}|Y_{n,k}|^2}}.$$

We write  $X_n = e^{\beta\sqrt{na_1}\xi}$ ,  $Y_n, X'_n, Y'_n, W_n$  for random variables having the same distribution as  $X_{n,k}, Y_{n,k}, X'_{n,k}, Y'_{n,k}, W_{n,k}$ . Note that by Lemma 4.2 (which holds locally uniformly in  $\beta \in \mathbb{C} \setminus \{0\}$ ),

$$(5.4) \quad z_n^2 := \text{Var } \mathcal{Z}_n(\beta) \sim N_{n,1} \mathbb{E}|X_n|^2 \mathbb{E}|Y_n|^2.$$

As we will see later, the conditions of the Lyapunov (and even Lindeberg) central limit theorem are not satisfied in the boundary case. Instead, we will use Theorem 3.20. In Steps 1–5 below, we will verify the conditions of Theorem 3.20 for the array  $\{z_n^{-1}W_{n,k} : 1 \leq k \leq N_{n,1}\}$ . The proof will be completed in Step 6.

STEP 1. We will verify the second condition of Theorem 3.20 by showing that for every  $\varepsilon > 0$ ,

$$(5.5) \quad \lim_{n \rightarrow \infty} N_{n,1} \text{Cov} \left( z_n^{-1}W_n \mathbb{1}_{|z_n^{-1}W_n| < \varepsilon} \right) = \frac{\Phi(u)}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{if } \tau \neq 0,$$

$$(5.6) \quad \lim_{n \rightarrow \infty} N_{n,1} \text{Cov} \left( z_n^{-1}W_n \mathbb{1}_{|z_n^{-1}W_n| < \varepsilon} \right) = \Phi(u) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{if } \tau = 0.$$

Here, we consider  $W_n$  as a vector with values in  $\mathbb{C} \equiv \mathbb{R}^2$  and  $\text{Cov}$  denotes the covariance matrix. To prove (5.5) and (5.6), it suffices to show that

$$(5.7) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E} \left[ |z_n^{-1}W_n|^2 \mathbb{1}_{|z_n^{-1}W_n| < \varepsilon} \right] = \Phi(u), \quad \text{for } \tau \in \mathbb{R},$$

$$(5.8) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E} \left[ (z_n^{-1}W_n)^2 \mathbb{1}_{|z_n^{-1}W_n| < \varepsilon} \right] = 0, \quad \text{for } \tau \neq 0,$$

$$(5.9) \quad \lim_{n \rightarrow \infty} \sqrt{N_{n,1}} \mathbb{E} \left[ |z_n^{-1}W_n| \mathbb{1}_{|z_n^{-1}W_n| < \varepsilon} \right] = 0, \quad \text{for } \tau \in \mathbb{R}.$$

We will prove (5.7), (5.8), (5.9) in Steps 2 and 3. Note in passing that (5.7) shows that the Lindeberg condition is not satisfied. (For the Lindeberg condition to hold, the limit in (5.7) should be 1). This is why we are using Theorem 3.20.

STEP 2. We prove (5.7) and (5.8). In view of (5.4), it is sufficient to show that for every  $\varepsilon > 0$ ,

$$(5.10) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ |X'_n Y'_n|^2 \mathbb{1}_{|X'_n Y'_n| < \varepsilon \sqrt{N_{n,1}}} \right] = \Phi(u), \quad \text{for } \tau \in \mathbb{R},$$

$$(5.11) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ (X'_n Y'_n)^2 \mathbb{1}_{|X'_n Y'_n| < \varepsilon \sqrt{N_{n,1}}} \right] = 0, \quad \text{for } \tau \neq 0.$$

Conditioning on  $Y'_n = y \in \mathbb{C}$ , using the total expectation formula and introducing the notation

$$(5.12) \quad f_n(y) = |y|^2 \mathbb{E} \left[ |X'_n|^2 \mathbb{1}_{|X'_n| < \varepsilon |y|^{-1} \sqrt{N_{n,1}}} \right],$$

$$(5.13) \quad \tilde{f}_n(y) = y^2 \mathbb{E} \left[ (X'_n)^2 \mathbb{1}_{|X'_n| < \varepsilon |y|^{-1} \sqrt{N_{n,1}}} \right],$$

we can write (5.10) and (5.11) as

$$(5.14) \quad \lim_{n \rightarrow \infty} \mathbb{E} f_n(Y'_n) = \Phi(u), \quad \text{for } \tau \in \mathbb{R},$$

$$(5.15) \quad \lim_{n \rightarrow \infty} \mathbb{E} \tilde{f}_n(Y'_n) = 0, \quad \text{for } \tau \neq 0.$$

The proof of (5.14) and (5.15) follows from Steps 2A, 2B, 2C below.

STEP 2A. We show that for every  $A > 1$ ,

$$(5.16) \quad \lim_{n \rightarrow \infty} \mathbb{E} [f_n(Y'_n) \mathbb{1}_{|Y'_n| \in [A^{-1}, A]}] = \Phi(u) \mathbb{E} [|Y'_n|^2 \mathbb{1}_{|Y'_n| \in [A^{-1}, A]}], \quad \text{for } \tau \in \mathbb{R},$$

$$(5.17) \quad \lim_{n \rightarrow \infty} \mathbb{E} [\tilde{f}_n(Y'_n) \mathbb{1}_{|Y'_n| \in [A^{-1}, A]}] = 0, \quad \text{for } \tau \neq 0.$$

It suffices to show that uniformly in  $|y| \in [A^{-1}, A]$ ,

$$(5.18) \quad \lim_{n \rightarrow \infty} f_n(y) = |y|^2 \Phi(u), \quad \text{for } \tau \in \mathbb{R},$$

$$(5.19) \quad \lim_{n \rightarrow \infty} \tilde{f}_n(y) = 0, \quad \text{for } \tau \neq 0.$$

Equivalently, we need to show that uniformly in  $c \in [A^{-1}, A]$  (where  $A > 1$  may be different now),

$$(5.20) \quad \lim_{n \rightarrow \infty} \mathbb{E} [|X'_n|^2 \mathbb{1}_{|X'_n| < c\sqrt{N_{n,1}}}] = \Phi(u), \quad \text{for } \tau \in \mathbb{R},$$

$$(5.21) \quad \lim_{n \rightarrow \infty} \mathbb{E} [(X'_n)^2 \mathbb{1}_{|X'_n| < c\sqrt{N_{n,1}}}] = 0, \quad \text{for } \tau \neq 0.$$

The inequality  $|X'_n| < c\sqrt{N_{n,1}}$  is equivalent to  $\xi < a_n$ , where

$$(5.22) \quad a_n = \frac{\frac{1}{2} \log N_{n,1} + \log c + \sigma^2 n a_1}{\sigma \sqrt{n a_1}}.$$

*Proof of (5.20).* By definition of  $X'_n$ , see (5.3), and Lemma 3.8 we have

$$\mathbb{E} [|X'_n|^2 \mathbb{1}_{|X'_n| < c\sqrt{N_{n,1}}}] = \mathbb{E} [e^{2\sigma\sqrt{na_1}\xi - 2\sigma^2 na_1} \mathbb{1}_{\xi < a_n}] = \Phi(a_n - 2\sigma\sqrt{na_1}).$$

Using (5.22), (1.1), (5.1), we obtain

$$(5.23) \quad a_n - 2\sigma\sqrt{na_1} = \frac{na_1(\frac{1}{4}\sigma_1^2 - \sigma^2) + o(\sqrt{n}) + \log c}{\sigma\sqrt{na_1}} = u + o(1).$$

This holds uniformly in  $c \in [A^{-1}, A]$ . We arrive at (5.20).

*Proof of (5.21).* By definition of  $X'_n$ , see (5.3), and Lemma 3.8 we have

$$\mathbb{E} [(X'_n)^2 \mathbb{1}_{|X'_n| < c\sqrt{N_{n,1}}}] = \mathbb{E} [e^{2\beta\sqrt{na_1}\xi - 2\sigma^2 na_1} \mathbb{1}_{\xi < a_n}] = e^{2(\beta^2 - \sigma^2)na_1} \Phi(a_n - 2\beta\sqrt{na_1}).$$

Now, we have  $\text{Re}(\beta^2 - \sigma^2) = -\tau^2 < 0$  since  $\tau \neq 0$ . For the same reason, we have  $\lim_{n \rightarrow \infty} \text{Im}(a_n - 2\beta\sqrt{na_1}) = \pm\infty$  (depending on the sign of  $\tau$ ) and it follows from (5.23) and Lemma 3.10 that

$$\lim_{n \rightarrow \infty} \Phi(a_n - 2\beta\sqrt{na_1}) = 0 \text{ or } 1.$$

This implies (5.21).

STEP 2B. We show that

$$(5.24) \quad \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} [f_n(Y'_n) \mathbb{1}_{|Y'_n| < A^{-1}}] = \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} |\mathbb{E} [\tilde{f}_n(Y'_n) \mathbb{1}_{|Y'_n| < A^{-1}}]| = 0.$$

Clearly,  $|\tilde{f}_n(y)| \leq f_n(y) \leq |y|^2$ , since  $\mathbb{E}|X'_n|^2 = 1$  by definition. It follows that  $\mathbb{E}[f_n(Y'_n)\mathbb{1}_{|Y'_n| < A^{-1}}] \leq A^{-2}$  and similarly with  $\tilde{f}_n$  instead of  $f_n$ . This implies (5.24).

STEP 2C. We show that

$$(5.25) \quad \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}[f_n(Y'_n)\mathbb{1}_{|Y'_n| > A}] = \lim_{A \rightarrow \infty} \limsup_{n \rightarrow \infty} |\mathbb{E}[\tilde{f}_n(Y'_n)\mathbb{1}_{|Y'_n| > A}]| = 0.$$

Note that  $Y_n$  is the analogue of  $\mathcal{Z}_n(\beta)$  with  $d-1$  levels. Since the smallest inverse critical temperature for  $Y_n$  is  $\sigma_2$  and  $\sigma < \frac{\sigma_2}{2}$ , there is  $p = 2 + \delta > 2$  such that  $\mathbb{E}|Y'_n|^{2+\delta} < C$  for all  $n \in \mathbb{N}$ ; see (4.13). From  $|\tilde{f}_n(y)| \leq f_n(y) \leq |y|^2$ , it follows that for all  $n \in \mathbb{N}$ ,

$$|\mathbb{E}[\tilde{f}_n(Y'_n)\mathbb{1}_{|Y'_n| > A}]| \leq \mathbb{E}[f_n(Y'_n)\mathbb{1}_{|Y'_n| > A}] \leq \mathbb{E}[|Y'_n|^2\mathbb{1}_{|Y'_n| > A}] \leq A^{-\delta}\mathbb{E}|Y'_n|^{2+\delta} < CA^{-\delta}.$$

This implies (5.25).

STEP 3. We prove (5.9). By (5.4), it suffices to show that for every  $\varepsilon > 0$ ,

$$(5.26) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ |X'_n Y'_n| \mathbb{1}_{|X'_n Y'_n| < \varepsilon \sqrt{N_{n,1}}} \right] = 0.$$

Take some  $A > 1$ . We can consider the cases  $|Y'_n| < A^{-1}$  and  $|Y'_n| \geq A^{-1}$  separately to obtain the estimate

$$\begin{aligned} \mathbb{E} \left[ |X'_n Y'_n| \mathbb{1}_{|X'_n Y'_n| < \varepsilon \sqrt{N_{n,1}}} \right] &\leq A^{-1} \mathbb{E}|X'_n| + \mathbb{E} \left[ |X'_n Y'_n| \mathbb{1}_{|X'_n| < A\varepsilon \sqrt{N_{n,1}}} \right] \\ &\leq A^{-1} + \mathbb{E} \left[ |X'_n| \mathbb{1}_{|X'_n| < A\varepsilon \sqrt{N_{n,1}}} \right], \end{aligned}$$

where in the second line we have used that  $\mathbb{E}|X'_n| \leq 1$  and  $\mathbb{E}|Y'_n| \leq 1$  since  $\mathbb{E}|X'_n|^2 = \mathbb{E}|Y'_n|^2 = 1$  by definition. Regarding the expectation on the right-hand side we obtain, by the definition of  $X'_n$  and Lemma 3.8,

$$\mathbb{E} \left[ |X'_n| \mathbb{1}_{|X'_n| < A\varepsilon \sqrt{N_{n,1}}} \right] = \mathbb{E}[e^{\sigma \sqrt{na_1} \xi - \sigma^2 na_1} \mathbb{1}_{\xi < a_n}] = e^{-\frac{1}{2} \sigma^2 na_1} \Phi(a_n - \sigma \sqrt{na_1}).$$

Here, we defined  $a_n$  as in (5.22) with  $c = A\varepsilon$ . Since we can estimate  $\Phi$  by 1, the right-hand side converges to 0, as  $n \rightarrow \infty$ . Combining everything together and letting  $A \rightarrow \infty$ , we obtain (5.26).

STEP 4. We show that, for every  $\varepsilon > 0$ ,

$$(5.27) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E} \left[ |z_n^{-1} W_n| \mathbb{1}_{|z_n^{-1} W_n| > \varepsilon} \right] = 0.$$

This statement will be needed to replace the truncated expectation by the usual one in Theorem 3.20. In view of (5.4), it suffices to show that

$$(5.28) \quad \lim_{n \rightarrow \infty} \sqrt{N_{n,1}} \mathbb{E} \left[ |X'_n Y'_n| \mathbb{1}_{|X'_n Y'_n| > \varepsilon \sqrt{N_{n,1}}} \right] = 0.$$

Introducing the function

$$g_n(y) = |y| \sqrt{N_{n,1}} \mathbb{E} \left[ |X'_n| \mathbb{1}_{|X'_n| > \varepsilon |y|^{-1} \sqrt{N_{n,1}}} \right],$$

where  $y \in \mathbb{C}$ , we can rewrite (5.28) in the following form:

$$(5.29) \quad \lim_{n \rightarrow \infty} \mathbb{E} g_n(Y'_n) = 0.$$

The proof of (5.29) will be provided in Steps 4A and 4B, below.

STEP 4A. Fix  $\delta \in (\frac{1}{2}, 1)$ . We will show that

$$(5.30) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ g_n(Y'_n) \mathbb{1}_{|Y'_n| < \varepsilon e^{\delta \sigma^2 n a_1}} \right] = 0.$$

Let  $y \in \mathbb{C}$  be such that  $|y| < \varepsilon e^{\delta \sigma^2 n a_1}$ . Defining  $a_n$  as in (5.22) with  $c = \varepsilon |y|^{-1}$ , we can write

$$g_n(y) = |y| \sqrt{N_{n,1}} e^{-\sigma^2 n a_1} \mathbb{E}[e^{\sigma \sqrt{n a_1} \xi} \mathbb{1}_{\xi > a_n}]$$

Arguing as in (5.23), we have  $a_n - \sigma \sqrt{n a_1} > \eta \sqrt{n}$ , for some  $\eta > 0$  and all sufficiently large  $n \in \mathbb{N}$ . Here, we used that  $\delta < 1$ . Hence, using Lemma 3.9, Part 1, inserting the value of  $a_n$ , see (5.22), and doing elementary transformations, we arrive at

$$\begin{aligned} g_n(y) &\leq \frac{C|y|}{\sqrt{n}} \sqrt{N_{n,1}} e^{-\sigma^2 n a_1} e^{\sigma \sqrt{n a_1} a_n - \frac{1}{2} a_n^2} \\ &\leq \frac{C|y|}{\sqrt{n}} e^{-\frac{1}{2\sigma^2 n a_1} \left( \left( \frac{1}{2} \log N_{n,1} - \sigma^2 n a_1 \right)^2 + \log^2 c + (\log N_{n,1})(\log c) \right)} \\ &\leq \frac{C}{\sqrt{n}} \varepsilon^{-\frac{\log N_{n,1}}{2\sigma^2 n a_1}} |y|^{1 + \frac{\log N_{n,1}}{2\sigma^2 n a_1}}, \end{aligned}$$

where in order to obtain the last inequality we used the non-negativity of the squares. By (1.1) and (5.1), we have  $\frac{1}{2} \log N_{n,1} = \sigma^2 n a_1 + O(\sqrt{n})$ . Take some  $p > 2$ . For sufficiently large  $n$ , we obtain the estimate

$$\mathbb{E} \left[ g_n(Y'_n) \mathbb{1}_{|Y'_n| < \varepsilon e^{\delta \sigma^2 n a_1}} \right] \leq \frac{C(\varepsilon)}{\sqrt{n}} \mathbb{E}|Y'_n \vee 1|^p \leq \frac{C(\varepsilon)}{\sqrt{n}} (\mathbb{E}|Y'_n|^p + 1).$$

By Lemma 4.4, the expectation on the right-hand side is bounded by a constant not depending on  $n$  if provided that  $p$  is sufficiently close to 2. This completes the proof of (5.30).

STEP 4B. In this step, we show that

$$(5.31) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[ g_n(Y'_n) \mathbb{1}_{|Y'_n| \geq \varepsilon e^{\delta \sigma^2 n a_1}} \right] = 0.$$

Using the definition of the function  $g_n$  and the inequality  $\mathbb{E}|X'_n| \leq 1$ , we obtain the estimate  $g_n(y) \leq |y| \sqrt{N_{n,1}}$  for all  $y \in \mathbb{C}$ . Hence,

$$\mathbb{E} \left[ g_n(Y'_n) \mathbb{1}_{|Y'_n| \geq \varepsilon e^{\delta \sigma^2 n a_1}} \right] \leq \sqrt{N_{n,1}} \mathbb{E} \left[ |Y'_n| \mathbb{1}_{|Y'_n| \geq \varepsilon e^{\delta \sigma^2 n a_1}} \right].$$

Using the fact that  $\mathbb{E}|Y'_n|^2 = 1$  and  $\frac{1}{2} \log N_{n,1} = \sigma^2 n a_1 + O(\sqrt{n})$ , see (1.1) and (5.1), we obtain that

$$\mathbb{E} \left[ g_n(Y'_n) \mathbb{1}_{|Y'_n| \geq \varepsilon e^{\delta \sigma^2 n a_1}} \right] \leq \varepsilon^{-1} e^{-\delta \sigma^2 n a_1} \cdot e^{\frac{1}{2} \sigma^2 n a_1 + O(\sqrt{n})},$$

which goes to 0 as  $n \rightarrow \infty$  since  $\delta > \frac{1}{2}$ . This proves (5.31).

STEP 5. It follows immediately from Step 4 that, for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{P}[|z_n^{-1} W_n| > \varepsilon] = 0.$$

This verifies the first condition of Theorem 3.20.

STEP 6. After we have verified the conditions of Theorem 3.20 for the array  $\{z_n^{-1} W_{n,k} : 1 \leq k \leq N_{n,1}\}$ , we can complete the proof of Theorem 2.10 as follows.



By Theorem 3.20, we have

$$(5.32) \quad z_n^{-1} \sum_{k=1}^{N_{n,1}} (W_{n,k} - \mathbb{E}[W_{n,k} \mathbb{1}_{|z_n^{-1} W_{n,k}| < 1}]) \xrightarrow[n \rightarrow \infty]{d} \begin{cases} N_{\mathbb{C}}(0, \Phi(u)), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0, \Phi(u)), & \text{if } \tau = 0. \end{cases}$$

Note that the covariance structure of the limiting distribution has been computed in (5.5) and (5.6). By Step 4, we can replace the truncated expectation  $\mathbb{E}[W_{n,k} \mathbb{1}_{|z_n^{-1} W_{n,k}| < 1}]$  in (5.32) by the usual expectation  $\mathbb{E}W_{n,k}$ . Recalling that  $\mathcal{Z}_n(\beta) = \sum_{k=1}^{N_{n,1}} W_{n,k}$ , we arrive at (5.2).

## 6. COVARIANCE STRUCTURE OF THE PARTITION FUNCTION

In this section, we prove asymptotic results on the covariance function of the random field  $\mathcal{Z}_n(\beta)$ . In particular, we prove Proposition 2.6.

**6.1. The variance of the partition function.** Fix some  $\beta \in \mathbb{C}$ . Recall that for  $0 \leq l \leq d$  we defined

$$(6.1) \quad b_l = \log \alpha + 2\sigma^2 a + \sum_{m=l+1}^d (\log \alpha_m - |\beta|^2 a_m).$$

We show that

$$(6.2) \quad \text{Var } \mathcal{Z}_n(\beta) \sim \begin{cases} e^{b_k n}, & \text{if } \frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}, \quad 1 \leq k \leq d, \\ e^{b_1 n}, & \text{if } 0 < |\beta| < \frac{\sigma_1}{\sqrt{2}}. \end{cases}$$

The boundary case  $|\beta| = \frac{\sigma_k}{\sqrt{2}}, 1 \leq k \leq d$ , will be considered in Remarks 6.6 and 6.8 below.

*Proof of (6.2).* Recall that  $a = a_1 + \dots + a_d$  is the variance of  $X_\varepsilon$ ,  $\varepsilon \in \mathbb{S}_n$ . Define also the “partial variances”  $A_{l,m} = a_l + \dots + a_m$  for  $1 \leq l \leq m \leq d$ . Let  $A_{l,m} = 0$  if  $l > m$ . Our aim is to compute the asymptotics of

$$\text{Var } \mathcal{Z}_n(\beta) = \mathbb{E}|\mathcal{Z}_n(\beta)|^2 - |\mathbb{E}\mathcal{Z}_n(\beta)|^2.$$

The subsequent estimates will be locally uniform in  $\beta$ .

**STEP 1.** Let us compute  $\mathbb{E}|\mathcal{Z}_n(\beta)|^2$  first. Fix some path  $\eta \in \mathbb{S}_n$  in the GREM tree, for example the “left-most” one  $\eta = (1, \dots, 1)$ . Then,

$$(6.3) \quad \mathbb{E}|\mathcal{Z}_n(\beta)|^2 = \mathbb{E}[\mathcal{Z}_n(\beta) \overline{\mathcal{Z}_n(\beta)}] = N_n \sum_{\varepsilon \in \mathbb{S}_n} \mathbb{E} e^{\sqrt{n}(\beta X_\eta + \bar{\beta} X_\varepsilon)} = \sum_{l=0}^d B_{n,l},$$

where in  $B_{n,l}$  we restrict the sum to the paths  $\varepsilon \in \mathbb{S}_n$  having exactly  $l$  common edges with  $\eta$ . That is,

$$(6.4) \quad \begin{aligned} B_{n,l} &= N_n \sum_{\varepsilon \in \mathbb{S}_n : l(\eta, \varepsilon) = l} \mathbb{E} e^{\sqrt{n}(\beta X_\eta + \bar{\beta} X_\varepsilon)} \\ &= N_n \cdot (N_{n,l+1} - 1) \cdot N_{n,l+2} \dots \cdot N_{n,d} \cdot e^{2\sigma^2 A_{1,l} n} e^{(\sigma^2 - \tau^2) A_{l+1,d} n}. \end{aligned}$$

Here,  $l(\eta, \varepsilon) = \min\{k \in \mathbb{N} : \varepsilon_k \neq 1\} - 1$  denotes the number of edges which are common to  $\varepsilon$  and  $\eta$  and we used the fact that

$$\text{Var}[\beta X_\eta + \bar{\beta} X_\varepsilon] = 4\sigma^2 A_{1,l} + 2(\sigma^2 - \tau^2) A_{l+1,d}.$$

Recall that  $\alpha = \alpha_1 \dots \alpha_d$  and  $N_n \sim \alpha^n$ . It follows that for every  $0 \leq l \leq d$ ,

$$(6.5) \quad B_{n,l} \sim \exp \left\{ n \left( \log \alpha + 2\sigma^2 a + \sum_{m=l+1}^d (\log \alpha_m - |\beta|^2 a_m) \right) \right\} \sim e^{b_l n}.$$

Assume now that  $\frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}$ , for some  $0 \leq k \leq d$ . (Recall that  $\sigma_0 = 0$  and  $\sigma_{d+1} = +\infty$ ). Then,  $b_k$  is strictly larger than all  $b_l$ 's with  $l \neq k$ . This is easily seen by noting that  $\log \alpha_m - |\beta|^2 a_m$  is negative for  $m \leq k$  and positive for  $m \geq k+1$ ; see (1.8). Hence, we obtain from (6.3) that

$$(6.6) \quad \mathbb{E}|\mathcal{Z}_n(\beta)|^2 \sim B_{n,k} \sim e^{b_k n}, \quad \text{for} \quad \frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}, \quad 0 \leq k \leq d.$$

STEP 2. We now show that

$$(6.7) \quad |\mathbb{E}\mathcal{Z}_n(\beta)|^2 = o(\mathbb{E}|\mathcal{Z}_n(\beta)|^2), \quad \text{for} \quad \frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}, \quad 1 \leq k \leq d.$$

Note that the case  $k = 0$  is excluded. By Proposition 2.5 we have

$$(6.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathbb{E}\mathcal{Z}_n(\beta)|^2 = 2 \log \alpha + (\sigma^2 - \tau^2)a = b_0.$$

On the other hand, it follows from the assumption  $k \neq 0$  that we have  $b_0 < b_k$ . Hence,

$$(6.9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}|\mathcal{Z}_n(\beta)|^2 = b_k > b_0.$$

Combining (6.8) and (6.9), we obtain that (6.7) holds.

STEP 3. From (6.6) and (6.7) we obtain that

$$(6.10) \quad \text{Var } \mathcal{Z}_n(\beta) \sim B_{n,k} \sim e^{b_k n}, \quad \text{for} \quad \frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}, \quad 1 \leq k \leq d.$$

STEP 4. Let us finally prove that

$$(6.11) \quad \text{Var } \mathcal{Z}_n(\beta) \sim B_{n,1} \sim e^{b_1 n}, \quad \text{for} \quad 0 < |\beta| < \frac{\sigma_1}{\sqrt{2}}.$$

Note that the variance is asymptotic to  $B_{n,1}$ , not  $B_{n,0}$ . Of course, we have  $\mathbb{E}|\mathcal{Z}_n(\beta)|^2 \sim B_{n,0}$  by (6.6), but we will show that the term  $B_{n,0}$  cancels in the formula for the variance. Namely, for  $B'_{n,0} := B_{n,0} - |\mathbb{E}\mathcal{Z}_n(\beta)|^2$  we have

$$(6.12) \quad \begin{aligned} B'_{n,0} &= N_n (N_{n,1} - 1) N_{n,2} \dots N_{n,d} e^{(\sigma^2 - \tau^2)an} - N_n^2 e^{(\sigma^2 - \tau^2)an} \\ &= -N_n^2 N_{n,1}^{-1} e^{(\sigma^2 - \tau^2)an}. \end{aligned}$$

It follows that

$$b'_0 := \lim_{n \rightarrow \infty} \frac{1}{n} \log |B'_{n,0}| = 2 \log \alpha - \log \alpha_1 + (\sigma^2 - \tau^2)a.$$

It follows from  $|\beta| < \frac{\sigma_1}{\sqrt{2}}$  that  $b_0 > b_1 > \dots > b_d$ . Therefore, since  $\beta \neq 0$ ,

$$(6.13) \quad b_1 = 2 \log \alpha + (\sigma^2 - \tau^2)a - (\log \alpha_1 - |\beta|^2 a_1) > \max\{b'_0, b_2, \dots, b_d\}.$$

It follows that the term  $B_{n,1}$  has larger order than  $B'_{n,0}, B_{n,2}, \dots, B_{n,d}$ . Hence,

$$\text{Var } \mathcal{Z}_n(\beta) = \mathbb{E}|\mathcal{Z}_n(\beta)|^2 - |\mathbb{E}\mathcal{Z}_n(\beta)|^2 = B'_{n,0} + \sum_{l=1}^d B_{n,l} \sim B_{n,1}.$$

Therefore, (6.11) holds.  $\square$

**6.2. Local covariance structure inside the rings.** In Sections 6.2 and 6.3, we look at the covariance of the partition function  $\mathcal{Z}_n(\beta)$  in a window of infinitesimal size near some fixed point  $\beta_* \in \mathbb{C}$ . We show that  $\mathcal{Z}_n(\beta)$  possesses some nontrivial limiting covariance structure inside this window. As in Section 6.1, there are phase transitions on the circles  $|\beta_*| = \frac{\sigma_k}{\sqrt{2}}$ ,  $1 \leq k \leq d$ ; see Figure 5, left.

Fix some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ . Define normalizing functions  $g_{n,1}(\beta_*; t), \dots, g_{n,d}(\beta_*; t)$ , where  $t \in \mathbb{C}$ , by

$$(6.14) \quad g_{n,l}(\beta_*; t) = \begin{cases} \frac{1}{2} \log N_{n,l} + a_l(\sqrt{n}\sigma_* + t)^2, & \text{if } |\beta_*| > \frac{\sigma_l}{\sqrt{2}}, \\ \log N_{n,l} + \frac{1}{2}a_l(\sqrt{n}\beta_* + t)^2, & \text{if } |\beta_*| < \frac{\sigma_l}{\sqrt{2}}. \end{cases}$$

Let

$$(6.15) \quad g_n(\beta_*; t) = g_{n,1}(\beta_*; t) + \dots + g_{n,d}(\beta_*; t).$$

Define stochastic processes  $\{Z_n(t) : t \in \mathbb{C}\}$  and  $\{Z_n^*(t) : t \in \mathbb{C}\}$  by

$$Z_n(t) = e^{-g_n(\beta_*; t)} \mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right), \quad Z_n^*(t) = Z_n(t) - \mathbb{E}Z_n(t).$$

**Proposition 6.1.** *Let  $\beta_* \in \mathbb{C} \setminus \mathbb{R}$  be such that for some  $1 \leq k \leq d$ ,*

$$(6.16) \quad \frac{\sigma_k}{\sqrt{2}} < |\beta_*| < \frac{\sigma_{k+1}}{\sqrt{2}}.$$

*Then, for every  $t_1, t_2, t \in \mathbb{C}$ ,*

$$(6.17) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] = \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] = e^{-\frac{1}{2}(a_1 + \dots + a_k)(t_1 - \bar{t}_2)^2},$$

$$(6.18) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) Z_n(t_2)] = \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) Z_n^*(t_2)] = 0,$$

$$(6.19) \quad \lim_{n \rightarrow \infty} \mathbb{E}Z_n(t) = 0.$$

*Proof.* To prove the proposition, we need to prove (6.19) and to show that

$$(6.20) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] = e^{-\frac{1}{2}(a_1 + \dots + a_k)(t_1 - \bar{t}_2)^2},$$

$$(6.21) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) Z_n(t_2)] = 0.$$

*Proof of (6.20).* Recall that  $\eta = (1, \dots, 1)$  is the left-most path in the GREM tree. Writing  $Z_n(t_1)$  and  $Z_n(t_2)$  as sums, see (1.7), and taking the products we obtain

$$\begin{aligned} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] &= e^{-g_n(\beta_*; t_1) - \overline{g_n(\beta_*; t_2)}} N_n \sum_{\varepsilon \in \mathbb{S}_n} \mathbb{E} e^{(\sqrt{n}\beta_* + t_1)X_\eta + (\sqrt{n}\bar{\beta}_* + \bar{t}_2)X_\varepsilon} \\ &= e^{-g_n(\beta_*; t_1) - \overline{g_n(\beta_*; t_2)}} \sum_{l=0}^d B_{n,l}(t_1, t_2), \end{aligned}$$

where in  $B_{n,l}(t_1, t_2)$  we take the sum over all paths  $\varepsilon \in \mathbb{S}_n$  in the GREM tree having exactly  $l$  edges in common with  $\eta$ , that is

$$\begin{aligned} (6.22) \quad B_{n,l}(t_1, t_2) &= N_n(N_{n,l+1} - 1)N_{n,l+2} \dots N_{n,d} \cdot \mathbb{E} \left[ e^{(\sqrt{n}\beta_* + t_1)X_\eta + (\sqrt{n}\bar{\beta}_* + \bar{t}_2)X_\varepsilon} \right] \\ &\sim N_n N_{n,l+1} \dots N_{n,d} \cdot e^{\frac{1}{2}(2\sqrt{n}\sigma_* + t_1 + \bar{t}_2)^2} A_{1,l} e^{\frac{1}{2}((\sqrt{n}\beta_* + t_1)^2 + (\sqrt{n}\bar{\beta}_* + \bar{t}_2)^2)A_{l+1,d}}. \end{aligned}$$

Note that  $B_{n,l}(t_1, t_2)$  differs from  $B_{n,l}$  by a factor  $e^{O(\sqrt{n})}$ ; see (6.4). By the same argument as in Section 6.1, condition (6.16) implies that  $B_{n,l}(t_1, t_2) = o(B_{n,k}(t_1, t_2))$  for all  $l \neq k$ . It follows that

$$\mathbb{E}[Z_n(t_1)\overline{Z_n(t_2)}] \sim e^{-g_n(\beta_*; t_1) - \overline{g_n(\beta_*; t_2)}} B_{n,k}(t_1, t_2) \sim e^{-\frac{1}{2}(a_1 + \dots + a_k)(t_1 - \bar{t}_2)^2},$$

where the last step follows by a simple calculation; see (6.14).

*Proof of (6.21).* We have

$$\begin{aligned} \mathbb{E}[Z_n(t_1)Z_n(t_2)] &= e^{-g_n(\beta_*; t_1) - g_n(\beta_*; t_2)} N_n \sum_{\varepsilon \in \mathbb{S}_n} \mathbb{E} e^{(\sqrt{n}\beta_* + t_1)X_\eta + (\sqrt{n}\beta_* + t_2)X_\varepsilon} \\ &= e^{-g_n(\beta_*; t_1) - g_n(\beta_*; t_2)} \sum_{l=0}^d C_{n,l}(t_1, t_2), \end{aligned}$$

where in  $C_{n,l}(t_1, t_2)$  we take the sum over all paths  $\varepsilon \in \mathbb{S}_n$  having exactly  $l$  edges in common with  $\eta$ , that is

$$\begin{aligned} (6.23) \quad C_{n,l}(t_1, t_2) &= N_n(N_{n,l+1} - 1)N_{n,l+2} \dots N_{n,d} \cdot \mathbb{E} \left[ e^{(\sqrt{n}\beta_* + t_1)X_\eta + (\sqrt{n}\beta_* + t_2)X_\varepsilon} \right] \\ &\sim N_n N_{n,l+1} \dots N_{n,d} \cdot e^{\frac{1}{2}(2\sqrt{n}\beta_* + t_1 + t_2)^2 A_{1,l}} e^{\frac{1}{2}((\sqrt{n}\beta_* + t_1)^2 + (\sqrt{n}\beta_* + t_2)^2) A_{l+1,d}}. \end{aligned}$$

Since  $\operatorname{Re}(\beta_*^2) < \sigma_*^2$  by the assumption  $\beta_* \notin \mathbb{R}$ , we see that  $C_{n,l}(t_1, t_2) = o(B_{n,l}(t_1, t_2))$  for every  $1 \leq l \leq d$ . For  $l = 0$ , we have  $A_{1,l} = 0$  and a weaker estimate  $C_{n,l}(t_1, t_2) = O(B_{n,l}(t_1, t_2))$ . In the proof of (6.20), we have shown that  $B_{n,l}(t_1, t_2) = o(B_{n,k}(t_1, t_2))$  for all  $l \neq k$ . Since the value  $l = 0$  is not optimal (by the assumption  $k \geq 1$ ), we obtain that  $C_{n,l}(t_1, t_2) = o(B_{n,k}(t_1, t_2))$  for all  $0 \leq l \leq d$ . This, together with the result of (6.20), yields (6.21).

*Proof of (6.19).* By Proposition 2.5, we have

$$\mathbb{E}Z_n(t) = N_n e^{-g_n(\beta_*; t)} e^{\frac{1}{2}(\sqrt{n}\beta_* + t)^2 a} = \prod_{l=1}^d (N_{n,l} e^{-g_{n,l}(\beta_*; t) + \frac{1}{2}n\beta_*^2 a_l + O(\sqrt{n})}).$$

It is easy to check using (6.14) that, for  $l \leq k$ , the corresponding term in the product is  $O(e^{-\varepsilon n})$ , for some  $\varepsilon > 0$ , whereas for  $l > k$  it is  $e^{O(\sqrt{n})}$  (in fact, 1). Since  $k \geq 1$ , we have at least one term of the form  $O(e^{-\varepsilon n})$ . It follows that the product converges to 0.  $\square$

Proposition 6.1 is not valid in the case  $k = 0$ . For this case, we need a slightly different normalization. Fix some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ . Define  $\hat{g}_n(\beta_*; t)$ , a modification of  $g_n(\beta_*; t)$ , by

$$(6.24) \quad \hat{g}_n(\beta_*; t) = \left( \frac{1}{2} \log N_{n,1} + a_1(\sqrt{n}\sigma_* + t)^2 \right) + \sum_{l=2}^d \left( \log N_{n,l} + \frac{1}{2}a_l(\sqrt{n}\beta_* + t)^2 \right).$$

Note that  $\hat{g}_n(\beta_*; t)$  differs from  $g_n(\beta_*; t)$  just by the way the first level is normalized. In the case  $k = 0$  we define the stochastic processes  $\{Z_n(t) : t \in \mathbb{C}\}$  and  $\{Z_n^*(t) : t \in \mathbb{C}\}$  by

$$Z_n(t) = e^{-\hat{g}_n(\beta_*; t)} \mathcal{Z}_n \left( \beta_* + \frac{t}{\sqrt{n}} \right), \quad Z_n^*(t) = Z_n(t) - \mathbb{E}Z_n(t).$$

**Proposition 6.2.** *Let  $\beta_* \in \mathbb{C} \setminus \mathbb{R}$  be such that  $|\beta_*| < \frac{\sigma_1}{\sqrt{2}}$ . Then, for every  $t_1, t_2, t \in \mathbb{C}$ ,*

$$(6.25) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] = e^{-\frac{1}{2}a_1(t_1 - \bar{t}_2)^2},$$

$$(6.26) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) Z_n^*(t_2)] = 0,$$

$$(6.27) \quad \lim_{n \rightarrow \infty} \mathbb{E}Z_n(t) = \infty.$$

**Remark 6.3.** It follows from (6.27) that (6.25) and (6.26) are not valid with  $Z_n^*$  replaced by  $Z_n$ .

*Proof of Proposition 6.2. Proof of (6.25).* In the same way as in the proof of Proposition 6.1, Eq. (6.20), we obtain that

$$\begin{aligned} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] &= e^{-\hat{g}_n(\beta_*, t_1) - \overline{\hat{g}_n(\beta_*, t_2)}} \sum_{l=0}^d B_{n,l}(t_1, t_2) \\ &\sim e^{-\hat{g}_n(\beta_*, t_1) - \overline{\hat{g}_n(\beta_*, t_2)}} B_{n,0}(t_1, t_2), \end{aligned}$$

where  $B_{n,l}(t_1, t_2)$  is the same as in that proof. However, we will show that the term  $B_{n,0}(t_1, t_2)$  cancels almost completely in the expression

$$\begin{aligned} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] &= \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] - \mathbb{E}[Z_n(t_1)] \overline{\mathbb{E}[Z_n(t_2)]} \\ &= e^{-\hat{g}_n(\beta_*, t_1) - \overline{\hat{g}_n(\beta_*, t_2)}} \left( B'_{n,0}(t_1, t_2) + \sum_{l=1}^d B_{n,l}(t_1, t_2) \right), \end{aligned}$$

where

$$B'_{n,0}(t_1, t_2) = B_{n,0}(t_1, t_2) - \mathbb{E}Z_n \left( \beta_* + \frac{t_1}{\sqrt{n}} \right) \overline{\mathbb{E}Z_n \left( \beta_* + \frac{t_2}{\sqrt{n}} \right)}.$$

Recalling the formula for  $B_{n,0}(t_1, t_2)$ , see (6.22), and using Proposition 2.5, we obtain that

$$\begin{aligned} (6.28) \quad B'_{n,0}(t_1, t_2) &= (N_n(N_{n,1} - 1)N_{n,2} \dots N_{n,d} - N_n^2) \cdot e^{\frac{1}{2}(\sqrt{n}\beta_* + t_1)^2 a + \frac{1}{2}(\sqrt{n}\bar{\beta}_* + \bar{t}_2)^2 a} \\ &= -N_n^2 N_{n,1}^{-1} e^{\frac{1}{2}(\sqrt{n}\beta_* + t_1)^2 a + \frac{1}{2}(\sqrt{n}\bar{\beta}_* + \bar{t}_2)^2 a}. \end{aligned}$$

Note that with  $B_{n,l}$  as in (6.4) and  $B'_{n,0}$  as in (6.12),

$$B_{n,l}(t_1, t_2) = B_{n,l} e^{O(\sqrt{n})}, \quad 0 \leq l \leq d, \quad B'_{n,0}(t_1, t_2) = B'_{n,0} e^{O(\sqrt{n})}.$$

Hence, by the argument from Section 6.1, see (6.13), we have

$$B_{n,l}(t_1, t_2) = o(B_{n,1}(t_1, t_2)), \quad 2 \leq l \leq d, \quad B'_{n,0}(t_1, t_2) = o(B_{n,1}(t_1, t_2)).$$

It follows that

$$\mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] \sim e^{-\hat{g}_n(\beta_*, t_1) - \overline{\hat{g}_n(\beta_*, t_2)}} B_{n,1}(t_1, t_2) \sim e^{-\frac{1}{2}a_1(t_1 - \bar{t}_2)^2},$$

where the last step follows by a simple calculation; see (6.22) and (6.24).

*Proof of (6.26).* In the same way as in the proof of Proposition 6.1, Eq. (6.21), we have

$$\mathbb{E}[Z_n(t_1) Z_n(t_2)] = e^{-\hat{g}_n(\beta_*, t_1) - \hat{g}_n(\beta_*, t_2)} \sum_{l=0}^d C_{n,l}(t_1, t_2),$$

where  $C_{n,l}(t_1, t_2)$  is the same as in that proof. Hence,

$$\begin{aligned}\mathbb{E}[Z_n^*(t_1)Z_n^*(t_2)] &= \mathbb{E}[Z_n(t_1)Z_n(t_2)] - \mathbb{E}[Z_n(t_1)]\mathbb{E}[Z_n(t_2)] \\ &= e^{-\hat{g}_n(\beta_*; t_1) - \hat{g}_n(\beta_*; t_2)} \left( C'_{n,0}(t_1, t_2) + \sum_{l=1}^d C_{n,l}(t_1, t_2) \right),\end{aligned}$$

where

$$C'_{n,0}(t_1, t_2) = C_{n,0}(t_1, t_2) - \mathbb{E}Z_n \left( \beta_* + \frac{t_1}{\sqrt{n}} \right) \mathbb{E}Z_n \left( \beta_* + \frac{t_2}{\sqrt{n}} \right).$$

Using the formula for  $C_{n,0}(t_1, t_2)$ , see (6.23), and Proposition 2.5, we obtain that

$$(6.29) \quad C'_{n,0}(t_1, t_2) = -N_n^2 N_{n,1}^{-1} e^{\frac{1}{2}(\beta_* \sqrt{n} + t_1)^2 a + \frac{1}{2}(\beta_* \sqrt{n} + t_2)^2 a}.$$

Note that  $\operatorname{Re}(\beta_*^2) < \sigma_*^2$  by the assumption  $\beta_* \notin \mathbb{R}$ . In the same way as in the proof of Proposition 6.1, we get that  $C_{n,l}(t_1, t_2) = o(B_{n,l}(t_1, t_2))$  for every  $0 \leq l \leq d$  (note that these terms differ from  $C_{n,l}$  and  $B_{n,l}$  by a factor of  $e^{O(\sqrt{n})}$ ). Additionally,  $C'_{n,0}(t_1, t_2) = o(B'_{n,0}(t_1, t_2))$ , compare (6.28) and (6.29). It follows that

$$\mathbb{E}[Z_n^*(t_1)Z_n^*(t_2)] = o(\mathbb{E}[Z_n^*(t_1)\overline{Z_n^*(t_2)}]) = o(1),$$

where the last step is by (6.25). This establishes (6.26).

*Proof of (6.27).* By Proposition 2.5, we have

$$\mathbb{E}Z_n(t) = N_n e^{-\hat{g}_n(\beta_*; t)} e^{\frac{1}{2}(\sqrt{n}\beta_* + t)^2 a} = e^{\frac{1}{2} \log N_{n,1} + \frac{1}{2}(\sqrt{n}\beta_* + t)^2 a_1 - (\sqrt{n}\sigma_* + t)^2 a_1}$$

The right-hand side goes to  $\infty$  by (1.1) and the assumption  $|\beta_*| < \frac{\sigma_1}{\sqrt{2}}$ .  $\square$

**Remark 6.4.** In Propositions 6.1 and 6.2, we did not consider the case of real  $\beta_*$ . If  $\beta_* \in \mathbb{R}$ , then the expressions for the limits of  $\mathbb{E}[Z_n(t_1)\overline{Z_n(t_2)}]$  and  $\mathbb{E}[Z_n^*(t_1)\overline{Z_n^*(t_2)}]$  remain the same, but the limits of  $\mathbb{E}[Z_n(t_1)Z_n(t_2)]$  and  $\mathbb{E}[Z_n^*(t_1)Z_n^*(t_2)]$  change, namely we have

$$\mathbb{E}[Z_n(t_1)Z_n(t_2)] = \mathbb{E}[Z_n(t_1)\overline{Z_n(\bar{t}_2)}], \quad \mathbb{E}[Z_n^*(t_1)Z_n^*(t_2)] = \mathbb{E}[Z_n^*(t_1)\overline{Z_n^*(\bar{t}_2)}].$$

The expressions for  $\mathbb{E}Z_n(t)$  remain the same.

**6.3. Local covariance structure on the boundary circles.** In this section, we compute the local covariance structure of the partition function  $\mathcal{Z}_n(\beta)$  in a small window around  $\beta_* \in \mathbb{C}$  such that  $|\beta_*| = \frac{\sigma_k}{\sqrt{2}}$ , for some  $1 \leq k \leq d$ . As we shall see, in order to obtain a non-trivial covariance function in the limit, we have to choose the linear size of the window to be of order  $\frac{1}{n}$ . The results of this section will be needed to prove Theorems 2.32 and 2.35 which describe the structure of the arc shaped “curves of zeros”.

Take some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ . Similarly to (6.1), we define, for  $0 \leq l \leq d$ ,

$$(6.30) \quad b_l = \log \alpha + 2\sigma_*^2 a + \sum_{m=l+1}^d (\log \alpha_m - |\beta_*|^2 a_m).$$

**Proposition 6.5.** *Let  $\beta_* \in \mathbb{C} \setminus \mathbb{R}$  be such that  $|\beta_*| = \frac{\sigma_k}{\sqrt{2}}$  for some  $2 \leq k \leq d$ . Define stochastic processes  $\{Z_n(t) : t \in \mathbb{C}\}$  and  $\{Z_n^*(t) : t \in \mathbb{C}\}$  by*

$$Z_n(t) = e^{-\frac{1}{2}b_k n} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right), \quad Z_n^*(t) = Z_n(t) - \mathbb{E}Z_n(t).$$

Then, for every  $t_1, t_2, t \in \mathbb{C}$ ,

(6.31)

$$\lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] = \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] = e^{t_1 \lambda_k + \bar{t}_2 \bar{\lambda}_k} + e^{t_1 \lambda_{k-1} + \bar{t}_2 \bar{\lambda}_{k-1}},$$

(6.32)

$$\lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) Z_n(t_2)] = \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) Z_n^*(t_2)] = 0,$$

(6.33)

$$\lim_{n \rightarrow \infty} \mathbb{E} Z_n(t) = 0,$$

where  $\lambda_l = 2\sigma_* A_{1,l} + \beta_* A_{l+1,d}$ ,  $1 \leq l \leq d$ .

**Remark 6.6.** In particular, we obtain that under the assumptions of Proposition 6.5,

$$\text{Var } \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) \sim e^{b_k n} (e^{2 \text{Re}(\lambda_k t)} + e^{2 \text{Re}(\lambda_{k-1} t)}).$$

*Proof.* To prove the proposition, we need to prove (6.33) and to show that

$$(6.34) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] = e^{t_1 \lambda_k + \bar{t}_2 \bar{\lambda}_k} + e^{t_1 \lambda_{k-1} + \bar{t}_2 \bar{\lambda}_{k-1}},$$

$$(6.35) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) Z_n(t_2)] = 0.$$

*Proof of (6.34).* Let  $\eta \in \mathbb{S}_n$  be some fixed path in the GREM tree, say  $\eta = (1, \dots, 1)$ . We have

$$\begin{aligned} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] &= e^{-nb_k} N_n \sum_{\varepsilon \in \mathbb{S}_n} \mathbb{E} e^{\sqrt{n}(\beta_* + \frac{t_1}{n})X_\eta + \sqrt{n}(\bar{\beta}_* + \frac{\bar{t}_2}{n})X_\varepsilon} \\ &= e^{-nb_k} \sum_{l=0}^d D_{n,l}(t_1, t_2), \end{aligned}$$

where in  $D_{n,l}(t_1, t_2)$  we take the sum over all paths  $\varepsilon \in \mathbb{S}_n$  in the GREM tree having exactly  $l$  edges in common with  $\eta$ , that is

$$\begin{aligned} D_{n,l}(t_1, t_2) &= N_n (N_{n,l+1} - 1) N_{n,l+2} \dots N_{n,d} \cdot \mathbb{E} e^{\sqrt{n}(\beta_* + \frac{t_1}{n})X_\eta + \sqrt{n}(\bar{\beta}_* + \frac{\bar{t}_2}{n})X_\varepsilon} \\ &\sim N_n N_{n,l+1} \dots N_{n,d} \cdot e^{\frac{n}{2}(2\sigma_* + \frac{1}{n}(t_1 + \bar{t}_2))^2 A_{1,l}} e^{\frac{n}{2}((\beta_* + \frac{t_1}{n})^2 + (\bar{\beta}_* + \frac{\bar{t}_2}{n})^2) A_{l+1,d}} \\ &\sim e^{nb_l} e^{2\sigma_* (t_1 + \bar{t}_2) A_{1,l}} e^{(\beta_* t_1 + \bar{\beta}_* \bar{t}_2) A_{l+1,d}}. \end{aligned}$$

The last step follows from (1.1) and (6.30). From the condition  $|\beta_*| = \frac{\sigma_k}{\sqrt{2}}$ , it follows that  $b_k = b_{k-1}$  and that  $b_l < b_k$  for  $l \notin \{k, k-1\}$ . This means that only the terms  $D_{n,k}(t_1, t_2)$  and  $D_{n,k-1}(t_1, t_2)$  are asymptotically relevant. Since

$$D_{n,k}(t_1, t_2) + D_{n,k-1}(t_1, t_2) \sim e^{nb_k} (e^{t_1 \lambda_k + \bar{t}_2 \bar{\lambda}_k} + e^{t_1 \lambda_{k-1} + \bar{t}_2 \bar{\lambda}_{k-1}}),$$

we arrive at (6.34).

*Proof of (6.35).* We have

$$\begin{aligned} \mathbb{E}[Z_n(t_1) Z_n(t_2)] &= e^{-nb_k} N_n \sum_{\varepsilon \in \mathbb{S}_n} \mathbb{E} e^{\sqrt{n}(\beta_* + \frac{t_1}{n})X_\eta + \sqrt{n}(\beta_* + \frac{t_2}{n})X_\varepsilon} \\ &= e^{-nb_k} \sum_{l=0}^d E_{n,l}(t_1, t_2), \end{aligned}$$

where in  $E_{n,l}(t_1, t_2)$  we take the sum over all paths  $\varepsilon \in \mathbb{S}_n$  having exactly  $l$  edges in common with  $\eta$ , that is

$$\begin{aligned} E_{n,l}(t_1, t_2) &= N_n(N_{n,l+1} - 1)N_{n,l+2} \dots N_{n,d} \cdot \mathbb{E} e^{\sqrt{n}(\beta_* + \frac{t_1}{n})X_\eta + \sqrt{n}(\beta_* + \frac{t_2}{n})X_\varepsilon} \\ &\sim N_n N_{n,l+1} \dots N_{n,d} \cdot e^{\frac{n}{2}(2\beta_* + \frac{1}{n}(t_1+t_2))^2 A_{1,l}} e^{\frac{n}{2}((\beta_* + \frac{t_1}{n})^2 + (\beta_* + \frac{t_2}{n})^2) A_{l+1,d}}. \end{aligned}$$

Since  $\operatorname{Re}(\beta_*^2) < \sigma_*^2$  by the assumption  $\beta_* \notin \mathbb{R}$ , we have  $E_{n,l}(t_1, t_2) = o(D_{n,l}(t_1, t_2))$  and hence  $E_{n,l}(t_1, t_2) = o(e^{nb_k})$  for all  $1 \leq l \leq d$ . For  $l = 0$ , this argumentation does not work since  $A_{1,0} = 0$ . Instead, for  $l = 0$ , we have

$$E_{n,0}(t_1, t_2) = N_n^2 e^{n\beta_*^2 a + O(1)} = o(e^{nb_k}),$$

where the last step holds since  $b_k > b_0 = 2 \log \alpha + (\sigma_*^2 - \tau_*^2)a$  for  $2 \leq k \leq d$ ; see (6.30). Note that this argument does fails for  $k = 1$ . (Which is the reason why we excluded the case  $k = 1$  in Proposition 6.5). Summarizing, we have shown that  $E_{n,l}(t_1, t_2) = o(e^{nb_k})$  for all  $0 \leq l \leq d$ . The proof of (6.35) is complete.

*Proof of (6.33).* We have

$$\mathbb{E} Z_n(t) = N_n e^{-\frac{1}{2}b_k n} e^{\frac{1}{2}n(\beta_* + \frac{t}{n})^2 a} = e^{\frac{n}{2} \sum_{m=1}^k (\log \alpha_m - |\beta_*|^2 a_m) + O(1)}.$$

If  $2 \leq k \leq d$ , then the sum in the exponent is strictly negative and (6.33) follows. Note that for  $k = 1$  the sum contains just one term and this term is 0. This is another reason why Proposition 6.5 is not valid for  $k = 1$ .  $\square$

The next proposition covers the case  $k = 1$  which was left open in Proposition 6.5.

**Proposition 6.7.** *Let  $\beta_* \in \mathbb{C} \setminus \mathbb{R}$  be such that  $|\beta_*| = \frac{\sigma_1}{\sqrt{2}}$ . Define the stochastic processes  $\{Z_n(t) : t \in \mathbb{C}\}$  and  $\{Z_n^*(t) : t \in \mathbb{C}\}$  by*

$$Z_n(t) = N_n^{-1} e^{-\frac{1}{2}\beta_*^2 a n} \mathcal{Z}_n\left(\beta_* + \frac{t}{n}\right), \quad Z_n^*(t) = Z_n(t) - \mathbb{E} Z_n(t).$$

Then, for every  $t_1, t_2, t \in \mathbb{C}$ ,

$$(6.36) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] = e^{t_1 \lambda_1 + \bar{t}_2 \bar{\lambda}_1},$$

$$(6.37) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) Z_n^*(t_2)] = 0,$$

$$(6.38) \quad \lim_{n \rightarrow \infty} \mathbb{E} Z_n(t) = e^{\beta_* t a} = e^{\lambda_0 t},$$

where  $\lambda_1 = 2\sigma_* a_1 + \beta_* A_{2,d}$  and  $\lambda_0 = \beta_* a$ .

**Remark 6.8.** In particular, we obtain that under the assumptions of Proposition 6.7,

$$\operatorname{Var} \mathcal{Z}_n\left(\beta_* + \frac{t}{n}\right) \sim e^{b_1 n} e^{2 \operatorname{Re}(\lambda_1 t)}.$$

To see this, note that by the assumption  $|\beta_*| = \frac{\sigma_1}{\sqrt{2}}$  and (1.1) we have

$$(6.39) \quad b_0 = b_1 = 2 \log \alpha + (\sigma_*^2 - \tau_*^2)a, \quad |N_n e^{\frac{1}{2}\beta_*^2 a n}| \sim e^{\frac{1}{2}b_1 n}.$$

*Proof.* To prove the proposition, we need to establish (6.38) and to show that

$$(6.40) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) \overline{Z_n(t_2)}] = e^{t_1 \lambda_1 + \bar{t}_2 \bar{\lambda}_1} + e^{(\beta_* t_1 + \bar{\beta}_* \bar{t}_2)a},$$

$$(6.41) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n(t_1) Z_n(t_2)] = e^{\beta_* (t_1 + t_2)a}.$$



*Proof of (6.38).* We have

$$\mathbb{E}Z_n(t) = e^{-\frac{1}{2}\beta_*^2 an} e^{\frac{1}{2}n(\beta_* + \frac{t}{n})^2 a} = e^{\beta_* ta + o(1)}.$$

Note for future reference that the convergence is locally uniform in  $t$ .

*Proof of (6.40).* Recall (6.39). In the same way as in the proof of (6.34), we obtain that

$$\mathbb{E}[Z_n(t_1)\overline{Z_n(t_2)}] \sim e^{-nb_1} \sum_{l=0}^d D_{n,l}(t_1, t_2),$$

where  $D_{n,l}(t_1, t_2)$  satisfies

$$D_{n,l}(t_1, t_2) \sim e^{nb_1} e^{2\sigma_*(t_1 + \bar{t}_2)A_{1,l}} e^{(\beta_* t_1 + \bar{\beta}_* \bar{t}_2)A_{l+1,d}}.$$

From the assumption  $|\beta_*| = \frac{\sigma_1}{\sqrt{2}}$ , it follows that  $b_0 = b_1$  and  $b_l < b_1$  for  $2 \leq l \leq d$ . Thus, only the terms  $D_{n,0}(t_1, t_2)$  and  $D_{n,1}(t_1, t_2)$  are asymptotically relevant. We have

$$D_{n,1}(t_1, t_2) + D_{n,0}(t_1, t_2) \sim e^{nb_1} (e^{t_1 \lambda_1 + \bar{t}_2 \bar{\lambda}_1} + e^{(\beta_* t_1 + \bar{\beta}_* \bar{t}_2)a}).$$

This yields (6.40).

*Proof of (6.41).* In the same way as in the proof of (6.35), we have

$$\mathbb{E}[Z_n(t_1)Z_n(t_2)] = N_n^{-2} e^{-\beta_*^2 an} \sum_{l=0}^d E_{n,l}(t_1, t_2),$$

where  $E_{n,l}(t_1, t_2)$  satisfies

$$E_{n,l}(t_1, t_2) \sim N_n N_{n,l+1} \cdots N_{n,d} \cdot e^{\frac{n}{2}(2\beta_* + \frac{1}{n}(t_1 + t_2))^2 A_{1,l}} e^{\frac{n}{2}((\beta_* + \frac{t_1}{n})^2 + (\beta_* + \frac{t_2}{n})^2)A_{l+1,d}}.$$

Since  $\operatorname{Re}(\beta_*^2) < \sigma_*^2$  by the assumption  $\beta_* \notin \mathbb{R}$ , we have that  $E_{n,l}(t_1, t_2) = o(D_{n,l}(t_1, t_2))$  for all  $1 \leq l \leq d$ . However, for  $l = 0$ , we have  $A_{1,l} = 0$  and

$$E_{n,0}(t_1, t_2) \sim N_n^2 e^{\frac{n}{2}((\beta_* + \frac{t_1}{n})^2 + (\beta_* + \frac{t_2}{n})^2)a} \sim N_n^2 e^{\beta_*^2 an} e^{\beta_*(t_1 + t_2)a}.$$

This yields (6.41).  $\square$

**Remark 6.9.** In Propositions 6.5 and 6.7, we left open the case of real  $\beta_*$ . If  $\beta_* \in \mathbb{R}$ , then the same considerations as in Remark 6.4 apply.

## 7. FUNCTIONAL CENTRAL LIMIT THEOREMS FOR $|\sigma| < \frac{\sigma_1}{2}$

**7.1. Statements of functional central limit theorems.** We use the notation  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ . The next theorem is a functional central limit theorem in phase  $F^k E^{d-k}$ , for  $1 \leq k \leq d$ . It is a particular case and the first step in the proof of Theorem 2.28. Recall that  $g_n(\beta_*; t)$  was defined in (6.14) and (6.15), Section 6.2.

**Theorem 7.1.** *Let  $\beta_* \in \mathbb{C}$  be such that  $|\sigma_*| < \frac{\sigma_1}{2}$  and  $\frac{\sigma_k}{\sqrt{2}} < |\beta_*| < \frac{\sigma_{k+1}}{\sqrt{2}}$  for some  $1 \leq k \leq d$ . (These requirements are equivalent to  $\beta_* \in F^k E^{d-k}$ ). Then, the following convergence holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$(7.1) \quad \left\{ e^{-g_n(\beta_*; t)} \mathcal{Z}_n \left( \beta_* + \frac{t}{\sqrt{n}} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ \mathbb{X}(\sqrt{A_{1,k}} t) : t \in \mathbb{C} \},$$

where  $\{ \mathbb{X}(t) : t \in \mathbb{C} \}$  is the plane Gaussian analytic function (2.27) and  $A_{1,k} = a_1 + \dots + a_k$ .

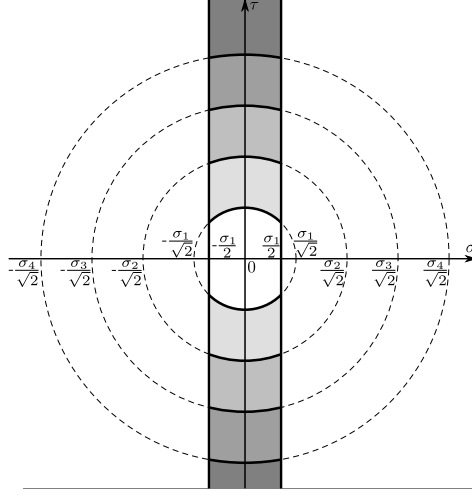


FIGURE 9. Domains in which the functional central limit theorems are valid.

Theorem 7.1 is not valid in the case  $k = 0$ . The reason is that for  $k = 0$  the expectation is larger than the fluctuations and so, an additional centering is needed to extract the fluctuations. For  $k = 0$ , we have the following result (which is a restatement of Theorem 2.20). Recall the definition of  $\hat{g}_n(\beta_*; t)$  from (6.24). Recall also that  $\mathcal{Z}_n^*(\beta_*) = \mathcal{Z}_n(\beta_*) - \mathbb{E}\mathcal{Z}_n(\beta_*)$ .

**Theorem 7.2.** *Let  $\beta_* \in \mathbb{C}$  be such that  $|\sigma_*| < \frac{\sigma_1}{2}$  and  $|\beta_*| < \frac{\sigma_1}{\sqrt{2}}$ . (This implies but is not equivalent to  $\beta_* \in E^d$ ). Then, the following convergence holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$(7.2) \quad \left\{ e^{-\hat{g}_n(\beta_*; t)} \mathcal{Z}_n^* \left( \beta_* + \frac{t}{\sqrt{n}} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ \mathbb{X}(\sqrt{a_1} t) : t \in \mathbb{C} \},$$

where  $\{ \mathbb{X}(t) : t \in \mathbb{C} \}$  is the plane Gaussian analytic function (2.27).

Note that we can replace  $\mathcal{Z}_n$  by  $\mathcal{Z}_n^*$  in Theorem 7.1, but we cannot replace  $\mathcal{Z}_n^*$  by  $\mathcal{Z}_n$  in Theorem 7.2.

Next, we are going to state a functional limit theorem for the boundary between the phases  $F^k E^{d-k}$  and  $F^{k-1} E^{d-k+1}$ , for  $2 \leq k \leq d$ . But first let us explain the idea. If we look at  $\mathcal{Z}_n$  locally at scale  $1/\sqrt{n}$  in phases  $F^k E^{d-k}$  and  $F^{k-1} E^{d-k+1}$ , we see essentially the Gaussian analytic functions  $\mathbb{X}(\sqrt{A_{1,k}} t)$  and  $\mathbb{X}(\sqrt{A_{1,k-1}} t)$ . In fact, it is convenient to think of  $\mathcal{Z}_n$  as of a weighted sum of all such Gaussian analytic functions over all  $k$ . However, the weights are such that in any phase *just one* Gaussian analytic function is dominating and all other functions are not visible in the limit. Now, if we look at  $\mathcal{Z}_n$  near the boundary of  $F^k E^{d-k}$  and  $F^{k-1} E^{d-k+1}$ , we see *two* Gaussian analytic functions *simultaneously*. It turns out that the right scale to look at in the boundary case is  $1/n$  (which is smaller than  $1/\sqrt{n}$ ). Hence, in fact, we see not two Gaussian analytic functions but rather just two Gaussian random variables,  $N'$  and  $N''$ , with some weights. Here is the exact statement.

**Theorem 7.3.** *Let  $\beta_* \in \mathbb{C}$  be such that  $|\sigma_*| < \frac{\sigma_1}{2}$  and  $|\beta_*| = \frac{\sigma_k}{\sqrt{2}}$  for some  $2 \leq k \leq d$ . Then, the following convergence holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$(7.3) \quad \left\{ e^{-\frac{1}{2}b_k n} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ e^{\lambda_k t} N' + e^{\lambda_{k-1} t} N'' : t \in \mathbb{C} \},$$

where  $N', N'' \sim N_{\mathbb{C}}(0, 1)$  are independent and  $\lambda_l = 2\sigma_* A_{1,l} + \beta_* A_{l+1,d}$ ,  $1 \leq l \leq d$ .

On the boundary between  $F^1 E^{d-1}$  and  $E^d$ , we have a slightly different functional central limit theorem. The reason is that in phase  $F^1 E^{d-1}$  the partition function  $\mathcal{Z}_n$  looks locally like a Gaussian analytic function, whereas in phase  $E^d$  (where the expectation dominates) the partition function looks locally like the expectation (plus Gaussian fluctuations which have smaller order of magnitude than the expectation). So, on the boundary between these two phases,  $\mathcal{Z}_n$  looks locally like a weighted sum of a Gaussian random variable  $N$  and a constant.

**Theorem 7.4.** *Let  $\beta_* \in \mathbb{C}$  be such that  $|\sigma_*| < \frac{\sigma_1}{2}$  and  $|\beta_*| = \frac{\sigma_1}{\sqrt{2}}$ . Then, the following convergence holds weakly on  $\mathcal{H}(\mathbb{C})$ :*

$$(7.4) \quad \left\{ N_n^{-1} e^{-\frac{1}{2}\beta_*^2 a n} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ e^{\lambda_1 t} N + e^{\lambda_0 t} : t \in \mathbb{C} \},$$

where  $N \sim N_{\mathbb{C}}(0, 1)$  and  $\lambda_1 = 2\sigma_* a_1 + \beta_* A_{2,d}$ ,  $\lambda_0 = \beta_* a$ .

**7.2. Proofs of functional central limit theorems.** All four theorems stated in Section 7.1 will be deduced from the following general result.

**Proposition 7.5.** *Fix some  $\beta_* \in \mathbb{C}$  such that  $|\sigma_*| < \frac{\sigma_1}{2}$ . Assume that  $c_n : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$  are deterministic analytic functions and  $q_n \in \mathbb{C}$  is a deterministic sequence such that the process  $Z_n^*(t) := c_n^{-1}(t) \mathcal{Z}_n^*(\beta_* + q_n t)$  has the property that for all  $t_1, t_2, t \in \mathbb{C}$ ,*

$$(7.5) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) \overline{Z_n^*(t_2)}] = \mathbb{E}[Z_\infty^*(t_1) \overline{Z_\infty^*(t_2)}],$$

$$(7.6) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Z_n^*(t_1) Z_n^*(t_2)] = \mathbb{E}[Z_\infty^*(t_1) Z_\infty^*(t_2)] = 0,$$

$$(7.7) \quad \text{Var } Z_n^*(t) \leq F(t),$$

where  $\{Z_\infty^*(t) : t \in \mathbb{C}\}$  is a zero-mean complex Gaussian process with sample paths in  $\mathcal{H}(\mathbb{C})$  and  $F : \mathbb{C} \rightarrow \mathbb{R}$  is a locally bounded function. Then, weakly on  $\mathcal{H}(\mathbb{C})$  it holds that

$$(7.8) \quad \{Z_n^*(t) : t \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \{Z_\infty^*(t) : t \in \mathbb{C}\}.$$

*Proof of Theorems 7.1, 7.2, 7.3, 7.4.* In Propositions 6.1, 6.2, 6.5, 6.7, we have shown that assumptions (7.5) and (7.6) are fulfilled with

- (1)  $q_n = 1/\sqrt{n}$ ,  $c_n(t) = e^{g_n(\beta_*, t)}$ ,  $Z_\infty^*(t) = \mathbb{X}(\sqrt{A_{1,k}} t)$  in Proposition 6.1.
- (2)  $q_n = 1/\sqrt{n}$ ,  $c_n(t) = e^{\hat{g}_n(\beta_*, t)}$ ,  $Z_\infty^*(t) = \mathbb{X}(\sqrt{a_1} t)$  in Proposition 6.2.
- (3)  $q_n = 1/n$ ,  $c_n(t) = e^{\frac{1}{2}b_k n}$ ,  $Z_\infty^*(t) = e^{t\lambda_k} N' + e^{t\lambda_{k-1}} N''$  in Proposition 6.5.
- (4)  $q_n = 1/n$ ,  $c_n(t) = N_n e^{\frac{1}{2}\beta_*^2 a n}$ ,  $Z_\infty^*(t) = e^{t\lambda_1} N$  in Proposition 6.7.

Condition (7.7) is satisfied because the statements of Propositions 6.1, 6.2, 6.5, 6.7 hold locally uniformly in  $t_1, t_2 \in \mathbb{C}$ , as it is easy to see from the proofs. Applying Proposition 7.5 we obtain Theorems 7.1, 7.2, 7.3, 7.4. In fact, in the fourth case, we obtain that

$$(7.9) \quad \left\{ N_n^{-1} e^{-\frac{1}{2}\beta_*^2 a n} \mathcal{Z}_n^* \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ e^{\lambda_1 t} N : t \in \mathbb{C} \}.$$

However, in Proposition 6.7, Eq. (6.38), we have shown that  $N_n^{-1}e^{-\frac{1}{2}\beta_*^2 an}\mathbb{E}Z_n(\beta_* + \frac{t}{n})$  converges to  $e^{\beta_* at}$  locally uniformly in  $t$  and hence, in  $\mathcal{H}(\mathbb{C})$ ; see the proof of (6.38). Together with (7.9), this yields (7.4).  $\square$

*Proof of Proposition 7.5.* We have the representation

$$Z_n^*(t) = \sum_{k=1}^{N_{n,1}} V_{n,k}^*(t),$$

where  $\{V_{n,k}^*(t) : t \in \mathbb{C}\}$  is a stochastic process defined by  $V_{n,k}^*(t) = V_{n,k}(t) - \mathbb{E}V_{n,k}(t)$  and

$$V_{n,k}(t) = c_n^{-1}(t) e^{\sqrt{na_1}(\beta_* + q_n t)\xi_k} \sum_{\varepsilon_2=1}^{N_{n,2}} \dots \sum_{\varepsilon_d=1}^{N_{n,d}} e^{\sqrt{n}(\beta_* + q_n t)(\sqrt{a_2}\xi_{k\varepsilon_2} + \dots + \sqrt{a_d}\xi_{k\varepsilon_2 \dots \varepsilon_d})}.$$

Note that for every  $n \in \mathbb{N}$  the processes  $\{V_{n,k}^*(t) : t \in \mathbb{C}\}$ ,  $1 \leq k \leq N_{n,1}$ , are independent by the definition of the GREM.

First, we show that (7.8) holds in the sense of weak convergence of finite-dimensional distributions. Pick some  $t_1, \dots, t_r \in \mathbb{C}$ . We show that the random vector  $\mathbf{S}_n^* := \{Z_n^*(t_i)\}_{i=1}^r$  converges to  $\mathbf{S}_\infty^* := \{Z_\infty^*(t_i)\}_{i=1}^r$  in distribution. We consider these  $r$ -dimensional complex random vectors as  $2r$ -dimensional real random vectors. To prove that  $\mathbf{S}_n^* \rightarrow \mathbf{S}_\infty^*$  in distribution, we will verify the conditions of Lyapunov's Theorem 3.18. By (7.5) and (7.6), the covariance matrix of  $\mathbf{S}_n^*$  converges to the covariance matrix of  $\mathbf{S}_\infty^*$ . This verifies the first condition of Theorem 3.18. It remains to verify the Lyapunov condition: For some  $p = 2 + \delta > 2$ ,

$$(7.10) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}|V_{n,1}^*(t_i)|^p = 0, \quad 1 \leq i \leq r.$$

Fix some  $1 \leq i \leq r$ . The random variable

$$\tilde{V}_n^* := \frac{c_n(t_i)}{\sqrt{\text{Var } Z_n(\beta_* + q_n t_i)}} V_{n,1}^*(t_i)$$

has the same distribution as the random variable  $z_n^{-1}W_n^*$  in Section 4 and hence, by (4.12), we obtain that  $N_{n,1}\mathbb{E}|\tilde{V}_n^*|^p$  converges to 0 as  $n \rightarrow \infty$  provided that  $\delta > 0$  is sufficiently small. Note that we have to insert  $\beta_* + q_n t_i$  instead of  $\beta$  in (4.12) but this causes no problems since (4.12) holds locally uniformly in the domain  $|\sigma| < \sigma_1/\sqrt{2p}$ . On the other hand, by (7.7) we have the estimate  $|V_{n,1}^*(t_i)| < C|\tilde{V}_n^*|$ . This completes the verification of (7.10).

Thus, we can apply Theorem 3.18 to obtain that  $\mathbf{S}_n^* \rightarrow \mathbf{S}_\infty^*$  in distribution. This means that the process  $\{Z_n^*(t) : t \in \mathbb{C}\}$  converges to  $\{Z_\infty^*(t) : t \in \mathbb{C}\}$  in the sense of finite-dimensional distributions. The fact that the sequence of processes  $\{Z_n^*(t) : t \in \mathbb{C}\}$ ,  $n \in \mathbb{N}$ , is tight in  $\mathcal{H}(\mathbb{C})$  follows from (7.7) and Proposition 3.12.  $\square$

## 8. MEROMORPHIC CONTINUATION OF THE POISSON CASCADE ZETA FUNCTION

**8.1. Uniform absolute convergence on compact sets: Proof of Theorem 2.12.** The first naïve attempt to prove Theorem 2.12 would be to try to demonstrate the absolute convergence of the integral

$$\int_0^\infty \dots \int_0^\infty x_1^{-z_1} \dots x_d^{-z_d} dx_1 \dots dx_d$$

for  $z \in \mathcal{D}$ . In view of Lemma 2.11, this would imply that  $\mathbb{E}\zeta_P(z) < \infty$ . However, this integral diverges because of the singularity which emerges if one of the variables

$x_1, \dots, x_d$  is close to 0. Following the method of [7], we will therefore introduce a subset  $F_\gamma(a)$  of  $\mathbb{R}_+^d = (0, \infty)^d$  in which all variables  $x_1, \dots, x_d$  are well-separated from 0. Then, we will show that the integral over this set converges. Over the complement of this set, the zeta sum can be reduced to a finite number of zeta functions of smaller dimension, and the induction can be applied. For  $d = 1$ , Theorem 2.12 follows from the fact that  $\lim_{n \rightarrow \infty} \frac{1}{n} P_n = 1$  a.s. by the law of large numbers. Henceforth, we assume that  $d \geq 2$ .

STEP 1. Fix some parameters  $\gamma_1 > \dots > \gamma_d > 0$ . Let the variables  $x = (x_1, \dots, x_d)$  and  $y = (y_1, \dots, y_d)$  take values in  $\mathbb{R}_+^d$  and be connected by the relations

$$(8.1) \quad y_1 = x_1^{\gamma_1}, \quad y_2 = x_1^{\gamma_1} x_2^{\gamma_2}, \quad \dots, \quad y_d = x_1^{\gamma_1} \dots x_d^{\gamma_d}.$$

The inverse transformation is given by

$$(8.2) \quad x_1 = y_1^{1/\gamma_1}, \quad x_2 = \left( \frac{y_2}{y_1} \right)^{1/\gamma_2}, \quad \dots, \quad x_d = \left( \frac{y_d}{y_{d-1}} \right)^{1/\gamma_d}.$$

We will often write  $dx$  and  $dy$  for  $dx_1 \dots dx_d$  and  $dy_1 \dots dy_d$ . Consider, for  $a > 0$ , the set

$$(8.3) \quad F_\gamma(a) = \{(x_1, \dots, x_d) \in \mathbb{R}_+^d : y_1 \geq a, \dots, y_d \geq a\}.$$

STEP 2. Let  $K \subset \mathcal{D}$  be a compact set. Consider a domain

$$(8.4) \quad \mathcal{D}_\gamma = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \frac{\operatorname{Re} z_1 - 1}{\gamma_1} > \dots > \frac{\operatorname{Re} z_d - 1}{\gamma_d} > 0 \right\} \subset \mathcal{D}.$$

We can find  $\gamma_1 > \dots > \gamma_d > 0$  such that  $K \subset \mathcal{D}_\gamma$ , just take all  $\gamma_i$ 's to be sufficiently close to 1. Moreover, it follows from (8.4) that we can find an  $\varepsilon > 0$  such that for all  $z \in K$ ,

$$(8.5) \quad \frac{\operatorname{Re} z_2}{\gamma_2} - \frac{\operatorname{Re} z_1}{\gamma_1} < \frac{1}{\gamma_2} - \frac{1}{\gamma_1} - \varepsilon, \quad \dots, \quad \frac{\operatorname{Re} z_d}{\gamma_d} - \frac{\operatorname{Re} z_{d-1}}{\gamma_{d-1}} < \frac{1}{\gamma_d} - \frac{1}{\gamma_{d-1}} - \varepsilon$$

and

$$(8.6) \quad \frac{\operatorname{Re} z_d}{\gamma_d} > \frac{1}{\gamma_d} + \varepsilon,$$

STEP 3. Let the set  $F = F_\gamma(1)$  be as in (8.3). Let  $x \in F$ . Then, for all  $z \in K$ ,

$$|x_1^{-z_1} \dots x_d^{-z_d}| = x_1^{\gamma_1 \left( -\frac{\operatorname{Re} z_1}{\gamma_1} \right)} \dots x_d^{\gamma_d \left( -\frac{\operatorname{Re} z_d}{\gamma_d} \right)} = y_d^{-\frac{\operatorname{Re} z_d}{\gamma_d}} y_{d-1}^{-\frac{\operatorname{Re} z_d}{\gamma_d} - \frac{\operatorname{Re} z_{d-1}}{\gamma_{d-1}}} \dots y_1^{-\frac{\operatorname{Re} z_2}{\gamma_2} - \frac{\operatorname{Re} z_1}{\gamma_1}}.$$

Since  $y_1 \geq 1, \dots, y_d \geq 1$  for  $x \in F$ , we obtain, by (8.5) and (8.6),

$$(8.7) \quad \varphi(x_1, \dots, x_d) := \sup_{z \in K} |x_1^{-z_1} \dots x_d^{-z_d}| \leq y_d^{-\frac{1}{\gamma_d} - \varepsilon} y_{d-1}^{-\frac{1}{\gamma_d} - \frac{1}{\gamma_{d-1}} - \varepsilon} \dots y_1^{-\frac{1}{\gamma_2} - \frac{1}{\gamma_1} - \varepsilon}.$$

Recall that  $\Pi$  is the Poisson cascade point process from Section 2.7. To prove Theorem 2.12, we need to show that

$$(8.8) \quad \sum_{x \in \Pi} \varphi(x) < +\infty \text{ a.s.}$$

Note that (8.8) is satisfied for  $d = 1$  since  $\lim_{n \rightarrow \infty} \frac{1}{n} P_n = 1$  a.s. by the law of large numbers. We can make the induction assumption that (8.8) holds in dimensions

$1, \dots, d-1$ . The proof of (8.8) will be complete after we have shown that in dimension  $d$ ,

$$(8.9) \quad S := \sum_{x \in \Pi \cap F} \varphi(x) < \infty \text{ a.s.} \quad \text{and} \quad R := \sum_{x \in \Pi \setminus F} \varphi(x) < \infty \text{ a.s.}$$

STEP 4. We prove that  $S < \infty$  a.s. The Jacobian of the transformation  $(x_1, \dots, x_d) \mapsto (y_1, \dots, y_d)$ , see (8.1), is given by  $\frac{dy}{dx} = \gamma_1 \dots \gamma_d \frac{y_1 \dots y_d}{x_1 \dots x_d}$ . Using this transformation, the estimate (8.7), and (8.2), we obtain that

$$\begin{aligned} \int_F \varphi(x) dx &\leq \int_1^\infty \dots \int_1^\infty y_d^{-\frac{1}{\gamma_d}-\varepsilon} y_{d-1}^{-\frac{1}{\gamma_d}-\frac{1}{\gamma_{d-1}}-\varepsilon} \dots y_1^{-\frac{1}{\gamma_2}-\frac{1}{\gamma_1}-\varepsilon} dx_1 \dots dx_d \\ &\leq \frac{1}{\gamma_1 \dots \gamma_d} \int_1^\infty \dots \int_1^\infty y_d^{-1-\varepsilon} \dots y_1^{-1-\varepsilon} dy_1 \dots dy_d. \end{aligned}$$

The integral on the right-hand side is finite. Hence,  $\int_F \varphi(x) dx < \infty$ . Since the intensity of the point process  $\Pi$  is the Lebesgue measure, see Lemma 2.11, we obtain that  $\mathbb{E}S < \infty$ . Hence,  $S$  is finite a.s.

STEP 5. We prove that  $R < \infty$  a.s. The idea is to reduce  $R$  to a finite number of zeta functions of dimension which is smaller than  $d$ . For  $m = 1, \dots, d$ , define a set

$$A_m = \{(x_1, \dots, x_m) \in \mathbb{R}_+^m : y_1 \geq 1, \dots, y_{m-1} \geq 1, y_m < 1\}.$$

Let  $E_m$  be the random set consisting of those indices  $(\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{N}^m$  for which the point  $(P_{\varepsilon_1}, P_{\varepsilon_1 \varepsilon_2}, \dots, P_{\varepsilon_1 \dots \varepsilon_m})$  is in  $A_m$ . We will show that  $E_m$  is finite with probability 1. In fact, we will even show that the expected number of elements in  $E_m$  is finite. Using the transformation  $(x_1, \dots, x_m) \mapsto (y_1, \dots, y_m)$ , see (8.1), we obtain

$$\begin{aligned} \int_{A_m} dx_1 \dots dx_m &= \frac{1}{\gamma_1 \dots \gamma_m} \int_{(1, \infty)^{m-1}} \int_0^1 y_1^{\frac{1}{\gamma_1}-\frac{1}{\gamma_2}-1} \dots y_{m-1}^{\frac{1}{\gamma_{m-1}}-\frac{1}{\gamma_m}-1} y_m^{\frac{1}{\gamma_m}-1} dy_1 \dots dy_m. \end{aligned}$$

The integral on the right-hand side is finite because  $\gamma_1 > \dots > \gamma_m > 0$ , thus proving that  $E_m$  is finite a.s. For every point  $x \in \mathbb{R}_+^d \setminus F$ , there exists a unique  $m = 1, \dots, d$  such that the projection of  $x$  onto the first  $m$  coordinates belongs to  $A_m$ . Hence, we have

$$(8.10) \quad R = \sum_{x \in \Pi \setminus F} \varphi(x) = \sum_{m=1}^d \sum_{(\varepsilon_1, \dots, \varepsilon_m) \in E_m} R_{\varepsilon_1 \dots \varepsilon_m},$$

where  $R_{\varepsilon_1 \dots \varepsilon_m}$  is a random variable defined by

$$R_{\varepsilon_1 \dots \varepsilon_m} = \sum_{\varepsilon_{m+1}, \dots, \varepsilon_d \in \mathbb{N}} \varphi(P_{\varepsilon_1}, \dots, P_{\varepsilon_1 \dots \varepsilon_d}).$$

Since, as we have shown, the sum on the right hand side of (8.10) involves a finite number of summands a.s., the proof of the a.s. finiteness of  $R$  would be complete if we could show that  $R_{\varepsilon_1, \dots, \varepsilon_m}$  is finite a.s. Let  $K_m$  be the projection of the compact

set  $K \subset \mathcal{D} \subset \mathbb{R}_+^d$  onto the last  $d - m$  coordinates. There is a constant  $C_{\varepsilon_1, \dots, \varepsilon_m}$  such that

$$R_{\varepsilon_1 \dots \varepsilon_m} \leq C_{\varepsilon_1, \dots, \varepsilon_m} \sum_{\varepsilon_{m+1}, \dots, \varepsilon_d \in \mathbb{N}} \max_{(z_{m+1}, \dots, z_d) \in K_m} |P_{\varepsilon_1 \dots \varepsilon_{m+1}}^{-z_{m+1}} \dots P_{\varepsilon_1 \dots \varepsilon_d}^{-z_d}|.$$

The sum on the right-hand side has the same structure as the sum  $\sum_{x \in \Pi} \varphi(x)$ , but in dimension  $d - m$ . Therefore, by the induction hypothesis, this sum is finite a.s., thus proving  $R_{\varepsilon_1 \dots \varepsilon_m}$  is finite a.s. Hence,  $R < \infty$  a.s.

**8.2. Meromorphic continuation of  $\zeta_P$ : Proof of Theorem 2.13.** This section is devoted to the proof of Theorem 2.13. The main step will be done in Proposition 8.2. We continue to use the notation of the previous section.

**Proposition 8.1.** *For  $z = (z_1, \dots, z_d) \in \mathcal{D}_\gamma$ , we have*  
(8.11)

$$I_\gamma(z; a) := \int_{F_\gamma(a)} x_1^{-z_1} \dots x_d^{-z_d} dx = \frac{a^{\frac{1-z_1}{\gamma_1}}}{\gamma_1 \dots \gamma_d} \left( \prod_{k=1}^{d-1} \frac{1}{\frac{z_k-1}{\gamma_k} - \frac{z_{k+1}-1}{\gamma_{k+1}}} \right) \frac{\gamma_d}{z_d - 1}.$$

The integral in (8.11) converges absolutely for  $z \in \mathcal{D}_\gamma$ .

*Proof.* We use induction on  $d$ . For  $d = 1$ , the identity reduces to the integral

$$\int_{a^{1/\gamma_1}}^{\infty} x_1^{-z_1} dx_1 = \frac{1}{z_1 - 1} a^{\frac{1-z_1}{\gamma_1}}.$$

Assume that the identity (8.11) is true for  $d - 1$  variables. We prove that it holds for  $d$  variables. We can write the conditions  $y_1 \geq a, \dots, y_d \geq a$  in the following form:

$$x_1 \geq a^{1/\gamma_1} \text{ and } x_2^{\gamma_2} \geq ax_1^{-\gamma_1}, \dots, x_2^{\gamma_2} \dots x_d^{\gamma_d} \geq ax_1^{-\gamma_1}.$$

Note that the conditions on  $x_2, \dots, x_d$  are of the same form as in  $F_\gamma(a)$ , but with  $d - 1$  variables and with  $ax_1^{-\gamma_1}$  instead of  $a$ . Therefore, we define a set

$$(8.12) \quad F_\gamma(a) = \{(x_2, \dots, x_d) \in \mathbb{R}_+^{d-1} : x_2^{\gamma_2} \geq a, \dots, x_2^{\gamma_2} \dots x_d^{\gamma_d} \geq a\}.$$

Using Fubini's theorem and then applying the induction assumption to the integral over the variables  $x_2, \dots, x_d$ , we obtain

$$\begin{aligned} I_\gamma(z; a) &= \int_{a^{1/\gamma_1}}^{\infty} \left( x_1^{-z_1} \int_{F_\gamma(ax_1^{-\gamma_1})} x_2^{-z_2} \dots x_d^{-z_d} dx_2 \dots dx_d \right) dx_1 \\ &= \frac{a^{\frac{1-z_2}{\gamma_2}}}{\gamma_2 \dots \gamma_d} \left( \prod_{k=2}^{d-1} \frac{1}{\frac{z_k-1}{\gamma_k} - \frac{z_{k+1}-1}{\gamma_{k+1}}} \right) \frac{\gamma_d}{z_d - 1} \int_{a^{1/\gamma_1}}^{\infty} x_1^{-z_1 - \gamma_1 \frac{1-z_2}{\gamma_2}} dx_1. \end{aligned}$$

Evaluation of the integral yields the desired formula (8.11). Note that the integral converges since  $\operatorname{Re}(z_1 + \gamma_1 \frac{1-z_2}{\gamma_2}) > 1$  by the assumption  $z \in \mathcal{D}_\gamma$ .  $\square$

**Proposition 8.2.** *Let  $\Pi$  be the Poisson cascade point process defined in Section 2.7. Fix  $a > 0$  and  $\gamma_1 > \dots > \gamma_d > 0$ . With probability 1, the function*

$$(8.13) \quad \zeta_P^*(z_1, \dots, z_d; a) := \sum_{x \in \Pi \cap F_\gamma(a)} x_1^{-z_1} \dots x_d^{-z_d} - I_\gamma(z_1, \dots, z_d; a),$$

defined originally on  $\mathcal{D}_\gamma$ , has a meromorphic continuation to the following larger domain:

$$\frac{1}{2}\mathcal{D}_\gamma = \{z \in \mathbb{C}^d : 2z \in \mathcal{D}_\gamma\}.$$

The function  $f_\gamma(z; a) := (z_d - 1)\zeta_P^*(z; a)$  is a.s. analytic on  $\frac{1}{2}\mathcal{D}_\gamma$ . For every  $z \in \frac{1}{2}\mathcal{D}_\gamma$ ,

$$(8.14) \quad \mathbb{E}f_\gamma(z; a) = 0, \quad \text{Var } f_\gamma(z; a) = a^{\frac{1-2\text{Re } z_1}{\gamma_1}} \text{Var } f_\gamma(z; 1) < \infty.$$

*Proof.* We use induction over the number of levels  $d$ . For  $d = 1$ , the proposition has been established in Theorem 2.6 of [25]; see also (2.17). Take some  $d \geq 2$  and assume that the statement of the proposition, including (8.14), holds in dimensions  $1, \dots, d-1$ . We prove that it holds in dimension  $d$ . The idea is to represent the  $d$ -variate function  $\zeta_P^*$  as a sum of the terms  $P_k^{-z_1}$  multiplied by independent copies of the  $(d-1)$ -variate function  $\zeta_P^*$ .

STEP 1: NOTATION. Take some  $T > a$ . Define  $F_\gamma(a, T) = F_\gamma(a) \cap \{y_1 \leq T^{\gamma_1}\}$ , a truncated version of the set  $F_\gamma(a)$ , by

$$F_\gamma(a, T) = \{(x_1, \dots, x_d) \in \mathbb{R}_+^d : a \leq y_1 \leq T^{\gamma_1}, y_2 \geq a, \dots, y_d \geq a\}.$$

Consider also  $I_\gamma(z; a, T)$ , a truncated version of the integral  $I_\gamma(z; a)$ :

$$I_\gamma(z; a, T) = \int_{F_\gamma(a, T)} x_1^{-z_1} \dots x_d^{-z_d} dx.$$

By Proposition 8.1, the integral defining  $I_\gamma(z; a, T)$  converges absolutely for  $z \in \mathcal{D}_\gamma$  and hence, defines an analytic function of  $z$  on  $\mathcal{D}_\gamma$ . An exact formula for  $I_\gamma(z; a, T)$  will be provided later; see (8.22). By Theorem 2.12, the following expression defines a random function of  $z$  which is with probability 1 analytic on  $\mathcal{D}_\gamma$ :

$$(8.15) \quad \zeta_P^*(z; a, T) := \sum_{x \in \Pi \cap F_\gamma(a, T)} x_1^{-z_1} \dots x_d^{-z_d} - I_\gamma(z; a, T).$$

STEP 2: MEROMORPHIC CONTINUATION OF  $\zeta_P^*(z; a, T)$ . Write  $\tilde{x} = (x_2, \dots, x_d)$ ,  $\tilde{z} = (z_2, \dots, z_d)$ , etc. For  $\varepsilon_1 \in \mathbb{N}$  let  $\tilde{\Pi}_{\varepsilon_1}$  be the point process on  $\mathbb{R}^{d-1}$  given by

$$\tilde{\Pi}_{\varepsilon_1} = \sum_{\tilde{\varepsilon} = (\varepsilon_2, \dots, \varepsilon_d) \in \mathbb{N}^{d-1}} \delta(P_{\varepsilon_1 \varepsilon_2}, \dots, P_{\varepsilon_1 \dots \varepsilon_d}).$$

In the definition of  $\Pi$ , the  $(d-1)$ -dimensional point process  $\tilde{\Pi}_{\varepsilon_1}$  is “attached” to the point  $P_{\varepsilon_1}$ . Define the random functions  $\tilde{\zeta}_1^*(z; a), \tilde{\zeta}_2^*(z; a), \dots$  by

$$(8.16) \quad \tilde{\zeta}_k^*(z; a) = \sum_{\tilde{x} \in \tilde{\Pi}_k \cap F_{\tilde{\gamma}}(a P_k^{-\gamma_1})} x_2^{-z_2} \dots x_d^{-z_d} - I_{\tilde{\gamma}}(\tilde{z}; a P_k^{-\gamma_1}), \quad z \in \mathcal{D}_\gamma.$$

Here,  $F_{\tilde{\gamma}}(a) \subset \mathbb{R}_+^{d-1}$  is the set defined in (8.12) and by Proposition 8.1,

$$(8.17) \quad I_{\tilde{\gamma}}(\tilde{z}; a) = \int_{F_{\tilde{\gamma}}(a)} x_2^{-z_2} \dots x_d^{-z_d} dx = \frac{a^{\frac{1-z_2}{\gamma_2}}}{\gamma_2 \dots \gamma_d} \left( \prod_{k=2}^{d-1} \frac{1}{\frac{z_k-1}{\gamma_k} - \frac{z_{k+1}-1}{\gamma_{k+1}}} \right) \frac{\gamma_d}{z_d - 1}.$$

Let  $\mathcal{A}$  be the  $\sigma$ -algebra generated by  $P_1, P_2, \dots$ . Note that conditionally on  $\mathcal{A}$ , the random functions  $\tilde{\zeta}_1^*(z; a), \tilde{\zeta}_2^*(z; a), \dots$  are independent. Also,

$$(8.18) \quad \tilde{\zeta}_k^*(z; a) \stackrel{d}{=} \zeta_P^*(z_2, \dots, z_d; a P_k^{-\gamma_1}) \text{ conditionally on } \mathcal{A}.$$

Due to the absolute convergence of the series in (8.15) for  $z \in \mathcal{D}_\gamma$ , we can change the order of summation and write

$$(8.19) \quad f_\gamma(z; a, T) := (z_d - 1)\zeta_P^*(z; a, T) = S_1(z; a, T) + S_2(z; a, T),$$



where

$$(8.20) \quad S_1(z; a, T) = (z_d - 1) \sum_{a \leq P_k^{\gamma_1} \leq T^{\gamma_1}} P_k^{-z_1} \tilde{\zeta}_k^*(z; a),$$

$$(8.21) \quad S_2(z; a, T) = (z_d - 1) \left( \sum_{a \leq P_k^{\gamma_1} \leq T^{\gamma_1}} P_k^{-z_1} I_{\tilde{\gamma}}(\tilde{z}; a P_k^{-\gamma_1}) - I_{\gamma}(z; a, T) \right).$$

This representation is valid for  $z \in \mathcal{D}_{\gamma}$ . However, by the induction assumption and (8.18), the function  $\tilde{f}_k(z; a) := (z_d - 1) \tilde{\zeta}_k^*(z; a)$  (and hence, the function  $S_1(z; a, T)$ ) has an analytic continuation to  $\frac{1}{2}\mathcal{D}_{\gamma}$ , with probability 1. Concerning the analytic continuation of  $S_2(z; a, T)$ , note that although the integral defining  $I_{\tilde{\gamma}}(\tilde{z}; a)$  in (8.17) may diverge for  $z \notin \mathcal{D}_{\gamma}$ , the expression on the right-hand side of (8.17), multiplied by  $z_d - 1$ , is an analytic function of  $z \in \frac{1}{2}\mathcal{D}_{\gamma}$ . The following formula, which is valid for  $z \in \mathcal{D}_{\gamma}$ ,

$$(8.22) \quad I_{\gamma}(z; a, T) = \int_{a^{1/\gamma_1}}^T x_1^{-z_1} I_{\tilde{\gamma}}(\tilde{z}; a x_1^{-\gamma_1}) dx_1 = I_{\tilde{\gamma}}(\tilde{z}; a) \int_{a^{1/\gamma_1}}^T x_1^{-z_1 - \gamma_1 \frac{1-z_2}{\gamma_2}} dx_1$$

yields an analytic continuation of  $(z_d - 1)I_{\gamma}(z; a, T)$  to the domain  $\frac{1}{2}\mathcal{D}_{\gamma}$ . Therefore,  $S_2(z; a, T)$  has analytic continuation to  $\frac{1}{2}\mathcal{D}_{\gamma}$ . Hence, the function  $f_{\gamma}(z; a, T)$  defined in (8.19) has an analytic continuation to  $\frac{1}{2}\mathcal{D}_{\gamma}$ , with probability 1.

**STEP 3: EXPECTATION AND VARIANCE OF  $S_1(z; a, T)$  AND  $S_2(z; a, T)$ .** By the result of Step 2, we can view  $f_{\gamma}(z; a, T) = (z_d - 1) \zeta_P^*(z; a, T)$  as a random element with values in the space  $\mathcal{H}(\frac{1}{2}\mathcal{D}_{\gamma})$ . We will now prove that the limit of  $f_{\gamma}(z; a, T)$  (in the sense of  $\mathcal{H}(\frac{1}{2}\mathcal{D}_{\gamma})$  and as  $T \rightarrow \infty$ ) exists a.s. Since for  $z \in \mathcal{D}_{\gamma}$  this limit coincides with  $f_{\gamma}(z; a) = (z_d - 1) \zeta_P^*(z; a)$ , we get the desired meromorphic continuation of  $\zeta_P^*(z; a)$ .

We need to compute the first two moments of  $S_1(z; a, T)$  and  $S_2(z; a, T)$  for  $z \in \frac{1}{2}\mathcal{D}_{\gamma}$ . Introduce the functions  $\tilde{f}_k(z; a) = (z_d - 1) \tilde{\zeta}_k^*(z; a)$ ,  $k \in \mathbb{N}$ . Recall that  $\mathcal{A}$  is the  $\sigma$ -algebra generated by the Poisson process  $P_1, P_2, \dots$ . By (8.18) and the induction assumption (8.14), we have

$$(8.23) \quad \mathbb{E}[\tilde{f}_k(z; a) | \mathcal{A}] = 0, \quad \text{Var}[\tilde{f}_k(z; a) | \mathcal{A}] = (a P_k^{-\gamma_1})^{\frac{1-2\text{Re } z_2}{\gamma_2}} \text{Var } f_{\tilde{\gamma}}(\tilde{z}; 1) \quad \text{a.s.}$$

Using (8.20), (8.23) and then the total expectation formula  $\mathbb{E}S = \mathbb{E}[\mathbb{E}[S | \mathcal{A}]]$ , we obtain that

$$(8.24) \quad \mathbb{E}[S_1(z; a, T) | \mathcal{A}] = 0 \text{ a.s.}, \quad \mathbb{E}S_1(z; a, T) = 0.$$

Using (8.21), (8.22), and the fact that  $S_2(z; a, T)$  is  $\mathcal{A}$ -measurable, we obtain that

$$(8.25) \quad \mathbb{E}S_2(z; a, T) = \mathbb{E}[S_1(z; a, T) \overline{S_2(z; a, T)}] = 0.$$

We now compute the variance of  $S_1(z; a, T)$ . Using (8.20) and the scaling property of the variance in (8.23), we obtain

$$\begin{aligned} \text{Var}[S_1(z; a, T) | \mathcal{A}] &= \sum_{a < P_k^{\gamma_1} < T^{\gamma_1}} P_k^{-2\text{Re } z_1} \text{Var}[\tilde{f}_k(z; a) | \mathcal{A}] \\ &= a^{\frac{1-2\text{Re } z_2}{\gamma_2}} \text{Var } f_{\tilde{\gamma}}(\tilde{z}; 1) \sum_{a < P_k^{\gamma_1} < T^{\gamma_1}} P_k^{-2\text{Re } z_1 - \gamma_1 \frac{1-2\text{Re } z_2}{\gamma_2}}. \end{aligned}$$

Using the formula for the total variance  $\text{Var } S = \mathbb{E} \text{Var}[S|\mathcal{A}] + \text{Var } \mathbb{E}[S|\mathcal{A}]$  and noting that the second term in it vanishes by (8.24), we get

$$\text{Var } S_1(z; a, T) = a^{\frac{1-2\text{Re } z_2}{\gamma_2}} \text{Var } f_{\tilde{\gamma}}(\tilde{z}; 1) \int_{a^{1/\gamma_1}}^T x_1^{-2\text{Re } z_1 - \gamma_1 \frac{1-2\text{Re } z_2}{\gamma_2}} dx_1.$$

Using the definition of  $\mathcal{D}_\gamma$ , see (8.4), it is easy to check that

$$(8.26) \quad 2\text{Re } z_1 + \gamma_1 \frac{1-2\text{Re } z_2}{\gamma_2} > 1 \text{ for } z \in \frac{1}{2}\mathcal{D}_\gamma.$$

Hence, the integral converges as  $T \rightarrow \infty$ . We obtain

$$(8.27) \quad \lim_{T \rightarrow +\infty} \text{Var } S_1(z; a, T) = \gamma_1^{-1} a^{\frac{1-2\text{Re } z_1}{\gamma_1}} \frac{\text{Var } f_{\tilde{\gamma}}(\tilde{z}; 1)}{\frac{1-2\text{Re } z_2}{\gamma_2} - \frac{1-2\text{Re } z_1}{\gamma_1}}.$$

We compute the variance of  $S_2(z; a, T)$ . Since  $P_1, P_2, \dots$  form a homogeneous Poisson process with intensity 1, the variance of the linear statistic  $\sum_{k \in \mathbb{N}} \varphi(P_k)$  is given by  $\int_0^\infty |\varphi(x_1)|^2 dx_1$ . Using (8.21) and the scaling property of  $I_{\tilde{\gamma}}(\tilde{z}; a)$  following from (8.17), we obtain

$$\begin{aligned} \text{Var } S_2(z; a, T) &= |z_d - 1|^2 \int_{a^{1/\gamma_1}}^T x_1^{-2\text{Re } z_1} |I_{\tilde{\gamma}}(\tilde{z}; ax_1^{-\gamma_1})|^2 dx_1 \\ &= a^{\frac{2-2\text{Re } z_2}{\gamma_2}} |(z_d - 1)I_{\tilde{\gamma}}(\tilde{z}; 1)|^2 \int_{a^{1/\gamma_1}}^T x_1^{-2\text{Re } z_1 - \gamma_1 \frac{2-2\text{Re } z_2}{\gamma_2}} dx_1. \end{aligned}$$

By (8.26), the integral converges as  $T \rightarrow +\infty$ . We have

$$(8.28) \quad \lim_{T \rightarrow +\infty} \text{Var } S_2(z; a, T) = \gamma_1^{-1} a^{\frac{1-2\text{Re } z_1}{\gamma_1}} \frac{|(z_d - 1)I_{\tilde{\gamma}}(\tilde{z}; 1)|^2}{\frac{2-2\text{Re } z_2}{\gamma_2} - \frac{2-2\text{Re } z_1}{\gamma_1}}.$$

**STEP 4: MEROMORPHIC CONTINUATION OF  $\zeta_P^*(z; a)$ .** We are in position to complete the proof of Proposition 8.2. Fix an arbitrary  $a > 0$  and some compact set  $K \subset \frac{1}{2}\mathcal{D}_\gamma$ . Consider  $S_1(z; a, T)$  and  $S_2(z; a, T)$  as stochastic processes indexed by  $T \in \mathbb{N}$  and taking values in the Banach space  $C(K)$  of continuous functions on  $K$ . By the properties of the Poisson process and (8.20), (8.21), both processes have independent (but not identically distributed) increments. Also, both processes have zero mean by (8.24) and (8.25). Hence, for every  $z \in K$ , both  $\{S_1(z; a, T)\}_{T \in \mathbb{N}}$  and  $\{S_2(z; a, T)\}_{T \in \mathbb{N}}$  are martingales. By (8.27) and (8.28), both martingales are bounded in  $L^2$  and hence, for every  $z \in K$  the sequences  $S_1(z; a, T)$  and  $S_2(z; a, T)$  converge as  $T \rightarrow \infty$  to some random variables, a.s. and in  $L^2$ . Hence, both sequences  $S_1(z; a, T)$  and  $S_2(z; a, T)$  (viewed as sequences of stochastic processes on  $K$ ) converge as  $T \rightarrow \infty$  to some limiting stochastic processes  $S_1(z; a)$  and  $S_2(z; a)$ , in the sense of finite-dimensional distributions. In fact, both sequences are tight by Proposition 3.12 with  $p = 2$ . The assumptions of Proposition 3.12 are fulfilled since  $\text{Var } S_1(z; a, T)$  and  $\text{Var } S_2(z; a, T)$  are increasing in  $T$  and can be bounded by the limits given in (8.27) and (8.28). Hence, both sequences  $S_1(z; a, T)$  and  $S_2(z; a, T)$  converge as  $T \rightarrow \infty$  weakly on  $C(K)$ . We will show that in fact, they converge even a.s. A classical theorem of Lévy states that partial sums of independent (not necessarily identically distributed)  $\mathbb{R}$ -valued random variables converge weakly if and only if they converge a.s. Itô and Nisio [23] extended this result to variables with values in a Banach space. Recalling that the sequences  $\{S_1(z; a, T)\}_{T \in \mathbb{N}}$  and  $\{S_2(z; a, T)\}_{T \in \mathbb{N}}$  have independent increments, we obtain that both  $S_1(z; a, T)$  and

$S_2(z; a, T)$  (considered as  $C(K)$ -valued random variables) converge a.s. as  $T \rightarrow \infty$ . Since the uniform limit of analytic functions is analytic, we obtain the desired analytic continuation of  $f_\gamma(z; a)$ . It follows from (8.19) and (8.24) that  $\mathbb{E}f_\gamma(z; a) = 0$ . The scaling property of the variance in (8.14) follows from (8.19), (8.25), (8.27), (8.28).  $\square$

*Completing the proof of Theorem 2.13.* We use induction over  $d$ . For  $d = 1$ , the statement has been established in [25]; see also (2.17). Take some  $d \geq 2$ . Assume that the statement is valid in dimensions  $1, \dots, d-1$ . Our aim is to prove that it holds in dimension  $d$ . Fix some  $a > 0$  and  $\gamma_1 > \dots > \gamma_d$ . Consider a domain  $F_\gamma(a)$  as in (8.3). Recalling (8.13), we have, for every  $z \in \mathcal{D}_\gamma$ , a representation

$$(8.29) \quad (z_d - 1)\zeta_P(z) = (z_d - 1)\zeta_P^*(z; a) + (z_d - 1)I_\gamma(z; a) + (z_d - 1)\zeta_P^\diamond(z; a),$$

where

$$\zeta_P^\diamond(z; a) = \sum_{x \in \Pi \setminus F_\gamma(a)} x_1^{-z_1} \dots x_d^{-z_d}.$$

This representation is valid for  $z \in \mathcal{D}_\gamma$  since we can interchange the order of summation in the definition of  $\zeta_P$  by the absolute convergence established in Theorem 2.12. However, by Propositions 8.2 and 8.1, the first two terms on the right-hand side of (8.29) have an analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$  with probability 1.

Let us now show that the function  $(z_d - 1)\zeta_P^\diamond(z; a)$  has an analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$ , with probability 1. For concreteness, take  $a = 1$ . Introduce the sets  $E_m$  as in the proof of Theorem 2.12, Step 5. For  $z \in \mathcal{D}_\gamma$ , we have a representation

$$(8.30) \quad \zeta_P^\diamond(z; a) = \sum_{m=1}^d \sum_{(\varepsilon_1, \dots, \varepsilon_m) \in E_m} P_{\varepsilon_1}^{-z_1} \dots P_{\varepsilon_1 \dots \varepsilon_m}^{-z_m} \zeta_{\varepsilon_1, \dots, \varepsilon_m}(z_{m+1}, \dots, z_d),$$

where

$$(8.31) \quad \zeta_{\varepsilon_1, \dots, \varepsilon_m}(z_{m+1}, \dots, z_d) = \sum_{\varepsilon_{m+1}, \dots, \varepsilon_d \in \mathbb{N}} P_{\varepsilon_1 \dots \varepsilon_{m+1}}^{-z_{m+1}} \dots P_{\varepsilon_1 \dots \varepsilon_d}^{-z_d}.$$

The functions  $\zeta_{\varepsilon_1, \dots, \varepsilon_m}(z_{m+1}, \dots, z_d)$  are  $(d-m)$ -variate analogues of the function  $(z_d - 1)\zeta_P(z)$  and hence, with probability 1 admit an analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$  by the induction assumption. Since the sets  $E_m$  are finite a.s. (as we have shown in the proof of Theorem 2.12, Step 5), we obtain the a.s. existence of an analytic continuation of  $(z_d - 1)\zeta_P^\diamond(z; a)$  to  $\frac{1}{2}\mathcal{D}_\gamma$ . The a.s. existence of the analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$  has been thus established for all three terms on the right-hand side of (8.29). This yields the desired analytic continuation of  $(z_d - 1)\zeta_P(z)$ .  $\square$

**8.3. A recursive formula for  $\zeta_P$ .** In this section, we will prove a formula allowing to represent the  $d$ -variate zeta function  $\zeta_P$  as a combination of infinitely many independent copies of the  $(d-1)$ -variate  $\zeta_P$ . Let  $\tilde{\mathcal{D}}$  be a  $(d-1)$ -dimensional analogue of the set  $\mathcal{D}$ , that is

$$\tilde{\mathcal{D}} = \{\tilde{z} = (z_2, \dots, z_d) \in \mathbb{C}^{d-1} : \operatorname{Re} z_2 > \dots > \operatorname{Re} z_d > 1\}.$$

Define independent random analytic functions  $\tilde{\zeta}_1, \tilde{\zeta}_2, \dots$  on  $\tilde{\mathcal{D}}$  by

$$(8.32) \quad \tilde{\zeta}_k(\tilde{z}) = \sum_{\tilde{x} \in \tilde{\Pi}_k} x_2^{-z_2} \dots x_d^{-z_d}, \quad k \in \mathbb{N}, \quad \tilde{z} \in \tilde{\mathcal{D}}.$$

Recall that for  $k \in \mathbb{N}$  we denote by  $\tilde{\Pi}_k$  the  $(d-1)$ -dimensional point process “attached” to the point  $P_k$  in the definition of the Poisson cascade point process  $\Pi$ , that is

$$\tilde{\Pi}_k = \sum_{\tilde{\varepsilon}=(\varepsilon_2, \dots, \varepsilon_d) \in \mathbb{N}^{d-1}} \delta(P_{k\varepsilon_2}, \dots, P_{k\varepsilon_2 \dots \varepsilon_d}).$$

For  $z \in \mathcal{D}$  (which implies that  $\tilde{z} \in \tilde{\mathcal{D}}$ ), we can interchange the order of summation in the definition of  $\zeta_P$  due to the absolute convergence established in Theorem 2.12. Hence,

$$(8.33) \quad \zeta_P(z) = \sum_{k=1}^{\infty} P_k^{-z_1} \tilde{\zeta}_k(\tilde{z}).$$

In the next proposition, we give an extension of (8.33) to  $\frac{1}{2}\mathcal{D}$ . Note that by Theorem 2.13, the functions  $\tilde{\zeta}_k(\tilde{z})$  (defined originally for  $\tilde{z} \in \tilde{\mathcal{D}}$ ) admit a meromorphic continuation to  $\frac{1}{2}\tilde{\mathcal{D}}$ , with probability 1.

**Proposition 8.3.** *Let  $d \geq 2$ . For  $T \in \mathbb{N}$  define  $\zeta_P(z; T)$ , a random meromorphic function on  $\frac{1}{2}\mathcal{D}$ , by*

$$(8.34) \quad \zeta_P(z; T) = \sum_{P_k \leq T} P_k^{-z_1} \tilde{\zeta}_k(\tilde{z}).$$

*Then, with probability 1,*

$$(8.35) \quad (z_d - 1)\zeta_P(z; T) \xrightarrow{T \rightarrow \infty} (z_d - 1)\zeta_P(z) \text{ on } \mathcal{H}\left(\frac{1}{2}\mathcal{D}\right).$$

**Remark 8.4.** It is important to stress that if we take  $d = 1$  and interpret  $\tilde{\zeta}_k(\tilde{z})$  as 1, then (8.35) does *not* hold since the series  $\sum_{k=1}^{\infty} P_k^{-z_1}$  converges in the half-plane  $\operatorname{Re} z_1 > 1$  only. In order to obtain an analogue of (8.35) for  $d = 1$ , one needs to subtract a regularizing term; see (2.17). Somewhat surprisingly, in the case  $d \geq 2$ , it is not necessary to subtract any regularizing terms from (8.34). The reason is that for  $d \geq 2$  the random variables  $\tilde{\zeta}_k(\tilde{z})$  are non-degenerate and it is known that multiplying the terms of a series by non-degenerate random variables may improve its convergence properties.

*Proof of Proposition 8.3.* First of all, it has been already observed above that (8.35) is valid for  $z \in \mathcal{D}$  by interchanging the order of summation. We will prove that the left-hand side of (8.35) converges as  $T \rightarrow \infty$  to *some* random analytic function in  $\mathcal{H}(\frac{1}{2}\mathcal{D})$ , with probability 1. The fact that the limiting function coincides with the right-hand side of (8.35), follows then by the uniqueness of the analytic continuation.

**STEP 1.** Fix some  $\gamma_1 > \dots > \gamma_d > 0$ . For  $z \in \mathcal{D}_\gamma$ , we can interchange the order of summation in the definition of  $\zeta_P(z; T)$  and hence, we have a representation

$$(8.36) \quad (z_d - 1)\zeta_P(z; T) = S_1(z; 1, T) + \bar{S}_2(z; 1, T) + S_3(z; 1, T),$$

where  $S_1(z; 1, T)$ ,  $\bar{S}_2(z; 1, T)$  and  $S_3(z; 1, T)$  are given by

$$\begin{aligned} S_1(z; 1, T) &= (z_d - 1) \sum_{1 \leq P_k \leq T} P_k^{-z_1} \tilde{\zeta}_k^*(z; 1), \\ \bar{S}_2(z; 1, T) &= (z_d - 1) I_{\tilde{\gamma}}(\tilde{z}; 1) \sum_{P_k \leq T} P_k^{-z_1 - \gamma_1 \frac{1-z_2}{\gamma_2}}, \\ S_3(z; 1, T) &= (z_d - 1) \sum_{x \in \Pi \setminus F_\gamma(1, T)} x_1^{-z_1} \dots x_d^{-z_d} \mathbb{1}_{x_1 \leq T}. \end{aligned}$$

Note that  $S_1(z; 1, T)$  is defined as in (8.20).

STEP 2. We have already shown in the proof of Theorem 2.13 that the function  $S_1(z; 1, T)$  admits an analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$  and that  $S_1(z; 1, T)$  converges, as  $T \rightarrow \infty$ , to a limiting random analytic function in  $\mathcal{H}(\frac{1}{2}\mathcal{D}_\gamma)$  with probability 1.

STEP 3. Let us consider  $\bar{S}_2(z; 1, T)$  next. Recall that the function  $(z_d - 1)I_{\tilde{\gamma}}(\tilde{z}; a)$ , defined as the right-hand side in (8.17) (but not as the integral there!), is an analytic function for  $z \in \frac{1}{2}\mathcal{D}_\gamma$ . This yields an analytic continuation of  $\bar{S}_2(z; 1, T)$  to  $\frac{1}{2}\mathcal{D}_\gamma$ . Let us prove the convergence of  $\bar{S}_2(z; 1, T)$ , as  $T \rightarrow \infty$ . By the definition of  $\mathcal{D}_\gamma$ , see (8.4), and the inequality  $\gamma_1 > \gamma_2 > 0$ , we have that, for every  $z \in \frac{1}{2}\mathcal{D}_\gamma$ ,

$$\operatorname{Re} \left( z_1 + \gamma_1 \frac{1 - z_2}{\gamma_2} \right) > 1.$$

Since  $\lim_{n \rightarrow \infty} \frac{1}{n} P_n = 1$  a.s. by the law of large numbers, we obtain that  $\bar{S}_2(z; 1, T)$  converges in  $\mathcal{H}(\frac{1}{2}\mathcal{D}_\gamma)$  with probability 1, as  $T \rightarrow \infty$ .

STEP 4. We will complete the proof by showing that  $S_3(z; 1, T)$  admits an analytic continuation to  $\frac{1}{2}\mathcal{D}_\gamma$  and converges to  $(z_d - 1)\zeta_P^\diamond(z; 1)$  in  $\mathcal{H}(\frac{1}{2}\mathcal{D}_\gamma)$  as  $T \rightarrow \infty$ , with probability 1. Recall (8.1) and (8.2). For  $m = 1, \dots, d$  and  $T \in \mathbb{N} \cup \{\infty\}$ , define a set

$$A_m(T) = \{(x_1, \dots, x_m) \in \mathbb{R}_+^m : 1 \leq y_1 < T^{\gamma_1}, y_2 \geq 1, \dots, y_{m-1} \geq 1, y_m < 1\}.$$

We interpret  $A_1(T)$  as  $(0, 1)$ . Let  $E_m(T)$  be the random set consisting of those indices  $(\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{N}^m$  for which the point  $(P_{\varepsilon_1}, P_{\varepsilon_1 \varepsilon_2}, \dots, P_{\varepsilon_1 \dots \varepsilon_m})$  is in  $A_m(T)$ . In the proof of Theorem 2.12, Step 5, we have shown that  $E_m = E_m(\infty)$  (and hence,  $E_m(T)$ ) is finite with probability 1. Now, for  $z \in \mathcal{D}_\gamma$ , we have a representation

$$S_3(z; 1, T) = (z_d - 1) \sum_{m=1}^d \sum_{(\varepsilon_1, \dots, \varepsilon_m) \in E_m(T)} P_{\varepsilon_1}^{-z_1} \dots P_{\varepsilon_1 \dots \varepsilon_m}^{-z_m} \zeta_{\varepsilon_1, \dots, \varepsilon_m}(z_{m+1}, \dots, z_d),$$

where  $\zeta_{\varepsilon_1, \dots, \varepsilon_m}(z_{m+1}, \dots, z_d)$  is defined as in (8.31). This provides an analytic continuation of  $S_3(z; 1, T)$  to  $\frac{1}{2}\mathcal{D}_\gamma$ . Since the  $\cup_{T \in \mathbb{N}} E_m(T) = E_m$  and  $E_m$  is a.s. finite, we must have  $E_m(T) = E_m$  for sufficiently large  $T$ , a.s. Hence,  $S_3(z; 1, T)$  coincides with  $(z_d - 1)\zeta_P^\diamond(z; 1)$  for sufficiently large  $T$ , a.s. This establishes the required statement.  $\square$

**8.4. Proof that  $\zeta_P(z)$  has no atoms.** The next proposition will be needed in the proof of Theorem 2.1.

**Proposition 8.5.** *For every  $z \in \frac{1}{2}\mathcal{D}$ , the random variable  $(z_d - 1)\zeta_P(z)$  has no atoms in  $\mathbb{C}$  except for  $d = 1$ ,  $z = 1$ .*

**Remark 8.6.** For  $d = 1$ ,  $z = z_1 = 1$ , we have  $(z - 1)\zeta_P(z) = 1$ , where the value is understood by continuity.

*Proof of Proposition 8.5.* We follow the method used in the proof of Lemma 3.10 in [25], where the case  $d = 1$  has been established. [Note that in this lemma the assumption  $\beta \neq 1$  is missing]. We may assume that  $d \geq 2$  and that the proposition has already been established in the setting of  $d - 1$  levels. Fix some  $z \in \frac{1}{2}\mathcal{D}$ . Given a random variable  $Y$  with values in  $\mathbb{C}$  write

$$Q(Y) = Q(Y; 0) = \sup_{w \in \mathbb{C}} \mathbb{P}[Y = w]$$

for the supremum over the probabilities of the atoms of  $Y$ . This is a special case of the concentration function; see (14.4) below. Let  $U$  and  $V$  be independent random variables with values in  $\mathbb{C}$ . It is easy to check that

(Q1)  $Q(U + V) \leq Q(U)$ ; see also (14.5) below.

(Q2) If  $U$  has no atoms and  $V$  has no atom at 0, then  $UV$  has no atoms.

By Proposition 8.3, for every  $T \in \mathbb{N}$  we have a representation  $(z_d - 1)\zeta_P(z) = U(T) + V(T)$ , where

$$U(T) = (z_d - 1)\zeta_P(z; T) = \sum_{k=1}^{\infty} P_k^{-z_1} \mathbb{1}_{P_k \in [0, T]} (z_d - 1)\tilde{\zeta}_k(\tilde{z})$$

and the random variables  $U(T)$  and  $V(T)$  are independent. We will prove that  $Q(U(T)) \leq e^{-T}$ . Then, by Property Q1, we would have  $Q((z_d - 1)\zeta_P(z)) \leq e^{-T}$  for every  $T \in \mathbb{N}$ . This would imply the statement of the proposition by letting  $T \rightarrow \infty$ . For  $m \in \mathbb{N}_0$ , let  $A_m(T)$  be the event which occurs if the interval  $[0, T]$  contains exactly  $m$  points of the form  $P_i$ ,  $i \in \mathbb{N}$ . Note that  $\mathbb{P}[A_0(T)] = e^{-T}$ . By the formula of the total probability, for every  $w \in \mathbb{C}$ ,

$$\mathbb{P}[U(T) = w] \leq e^{-T} + \sum_{m=1}^{\infty} \mathbb{P}[U(T) = w | A_m(T)].$$

Conditioned on  $A_m(T)$ , the random variables  $P_1, \dots, P_m$  have the same joint law as the increasing order statistics of the i.i.d. random variables  $\eta_1, \dots, \eta_m$  distributed uniformly on  $[0, T]$ . Therefore, by Property Q1, for every  $m \in \mathbb{N}$ ,

$$\mathbb{P}[U(T) = w | A_m(T)] \leq Q\left(\sum_{k=1}^m \eta_k^{-z_1} (z_d - 1)\tilde{\zeta}_k(\tilde{z})\right) \leq Q(\eta_1^{-z_1} (z_d - 1)\tilde{\zeta}_1(\tilde{z})) = 0.$$

The last step follows by Property Q2 from the fact that  $\eta_1^{-z_1}$  has no atoms and  $(z_d - 1)\tilde{\zeta}_1(\tilde{z})$  has no atom 0 by the induction assumption.  $\square$

A random vector with values in  $\mathbb{R}^m$  is called *full* if its distribution is not concentrated on some proper affine subspace of  $\mathbb{R}^m$ . The next proposition will be needed in the proof of Proposition 2.16.

**Proposition 8.7.** *If  $z \in \frac{1}{2}\mathcal{D} \setminus \mathbb{R}^d$ , then  $(z_d - 1)\zeta_P(z)$  is full in  $\mathbb{C} \equiv \mathbb{R}^2$ .*

*Proof.* Let  $U$  and  $V$  be independent random variables with values in  $\mathbb{C} \equiv \mathbb{R}^2$ . The following statements are easy to verify:

(F1) If  $U$  is full, then  $U + V$  is full.

(F2) If  $U$  is full and  $V$  has no atom at 0, then  $UV$  is full.

(F3) If  $A$  is an event with  $\mathbb{P}[A] > 0$  and the conditional distribution of  $U$  given  $A$  is full, then the unconditional distribution of  $U$  is full as well.

Assume first that  $z_1 \notin \mathbb{R}$ . We have a representation  $(z_d - 1)\zeta_P(z) = U + V$ , where  $U$  and  $V$  are independent random variables and  $U = \sum_{k=1}^{\infty} P_k^{-z_1} \mathbb{1}_{P_k \leq 1} (z_d - 1)\tilde{\zeta}_k(\tilde{z})$ . Indeed, for  $d = 1$  this (with  $\tilde{\zeta}_k(\tilde{z}) = 1$ ) follows from (2.17), whereas for  $d \geq 2$  the statement follows from Proposition 8.3. We will show that  $U$  is full. By Property F1, this would imply that  $U + V$  is full as well. Let  $A = \{P_1 \leq 1 < P_2\}$  be the event which occurs if the interval  $[0, 1]$  contains exactly 1 point of the form  $P_i$ ,  $i \in \mathbb{N}$ . The probability of  $A$  is equal to  $e^{-1}$  and hence is strictly positive. Since  $z_1 \notin \mathbb{R}$ , the conditional law of  $P_1^{-z_1}$  given  $A$  is full. Also,  $(z_d - 1)\tilde{\zeta}_1(\tilde{z})$  has no atom at 0 by Proposition 8.5. By Property F2, the conditional law of  $P_1^{-z_1}(z_d - 1)\tilde{\zeta}_1(\tilde{z})$  given  $A$  is full. On the event  $A$  this random variable is equal to  $U$ . By Property F3 the law of  $U$  is full. By Property F1 the law of  $U + V = (z_d - 1)\zeta_P(z)$  is full as well. This completes the proof in the case  $z_1 \notin \mathbb{R}$ .

Assume, by induction, that the statement of Proposition 8.7 holds in the setting of  $d - 1$  levels. Note that the basis of induction (that is, the case  $d = 1$ ,  $\operatorname{Re} z_1 > \frac{1}{2}$ ,  $z_1 \notin \mathbb{R}$ ) has been verified above. We prove that the proposition holds in the setting of  $d$  levels. We may assume that  $z_1 \in \mathbb{R}$  since we have already considered the case  $z_1 \notin \mathbb{R}$ . We use the same notation for  $U, V, A$  as above. By Property F1 it suffices to show that  $U$  is full. Since at least one coordinate among  $z_2, \dots, z_d$  is not real, the law of  $(z_d - 1)\tilde{\zeta}_1(\tilde{z})$  is full by the induction assumption. This random variable is independent from  $A$ , so its law remains full conditionally on  $A$ . Also, conditionally on  $A$ , the random variable  $P_1^{-z_1}$  has no atom at 0. By Property F2, the law of  $P_1^{-z_1}(z_d - 1)\tilde{\zeta}_1(\tilde{z})$ , conditionally on  $A$ , is full. Since this law coincides with the conditional law of  $U$  given  $A$ , we obtain that the unconditional law of  $U$  is full by Property F3.  $\square$

Let us finally mention a lemma which will be useful in Section 14.

**Lemma 8.8.** *Let  $N$  be a random variable with values in  $\mathbb{C}$  and having no atoms. Also, let  $(S_1, S_2)$  be a random vector with values in  $\mathbb{C}^2$  which is independent of  $N$  and such that  $S_1$  has no atom at 0. Then, the random variable  $S_1 N + S_2$  has no atoms.*

*Proof.* Let  $\mu$  be the distribution of  $(S_1, S_2)$  on  $\mathbb{C}^2$ . Then,  $\mu(\{0\} \times \mathbb{C}) = 0$ . By the convolution formula, for every  $w \in \mathbb{C}$  we have

$$\mathbb{P}[S_1 N + S_2 = w] = \int_{(\mathbb{C} \setminus \{0\}) \times \mathbb{C}} \mathbb{P}[s_1 N + s_2 = w] \mu(ds_1, ds_2).$$

To complete the proof, note that since  $N$  has no atoms, we have  $\mathbb{P}[s_1 N + s_2 = w] = 0$ , for every  $s_1 \in \mathbb{C} \setminus \{0\}$  and  $s_2 \in \mathbb{C}$ .  $\square$

### 8.5. Operator stability and moments of $\zeta_P$ : Proof of Propositions 2.15 and 2.16.

*Proof of Proposition 2.15.* The idea of the proof is the same as in Proposition 1.4.1 of [42]. Let  $d \geq 2$ . By Proposition 8.3, every function  $(z_d - 1)\zeta_P^{(j)}(z)$  can be represented as an a.s. limit of  $(z_d - 1)\zeta_P^{(j)}(z; T)$  as  $T \rightarrow \infty$ , where

$$(z_d - 1)\zeta_P^{(j)}(z; T) = \sum_{P_{k,j} \leq T} P_{k,j}^{-z_1} (z_d - 1)\tilde{\zeta}_{k,j}(\tilde{z}).$$

Here,  $\sum_{k=1}^{\infty} \delta(P_{k,j})$ ,  $1 \leq j \leq m$ , are independent unit intensity Poisson point processes on  $(0, \infty)$  and independently,  $(z_d - 1)\tilde{\zeta}_{k,j}(\tilde{z})$ ,  $k \in \mathbb{N}$ ,  $1 \leq j \leq m$ , are independent copies of the random analytic function  $(z_d - 1)\zeta_P(\tilde{z})$ . Then,

$$\begin{aligned} \sum_{j=1}^m (z_d - 1)\zeta_P^{(j)}(z; T) &= m^{z_1} \sum_{j=1}^m \sum_{mP_{k,j} \leq mT} (mP_{k,j})^{-z_1} (z_d - 1)\tilde{\zeta}_{k,j}(\tilde{z}) \\ &\stackrel{d}{=} m^{z_1} (z_d - 1)\zeta_P(z; mT) \end{aligned}$$

because  $\sum_{k=1}^{\infty} \sum_{j=1}^m \delta(mP_{k,j})$  is a Poisson point process on  $(0, \infty)$  with intensity 1. Letting  $T \rightarrow \infty$  yields the required statement. For  $d = 1$ , the proof is similar.  $\square$

*Proof of Proposition 2.16.* If  $z_k \in \mathbb{R}$ , for all  $1 \leq k \leq d$ , then the distribution of  $(z_d - 1)\zeta_P(z)$  is real stable with index  $\alpha := \frac{1}{z_1}$ . It is also non-degenerate by Proposition 8.5, except where  $d = 1$ ,  $z = 1$ . It is well-known that a non-degenerate  $\alpha$ -stable distributions has finite absolute moments of order  $p < \alpha$  and that the absolute moments of order  $p \geq \alpha$  are infinite; see [42, Property 1.2.16].

Henceforth, we may assume that  $z \in \frac{1}{2}\mathcal{D} \setminus \mathbb{R}^d$ . Consider the map  $w \mapsto m^{z_1}w$ ,  $w \in \mathbb{C}$ , as a linear operator on  $\mathbb{C} \equiv \mathbb{R}^2$ . In the basis  $\{1, i\}$ , this operator can be written as  $m^B$ , where  $B$  is the matrix

$$B = \begin{pmatrix} \operatorname{Re} z_1 & -\operatorname{Im} z_1 \\ \operatorname{Im} z_1 & \operatorname{Re} z_1 \end{pmatrix}.$$

By Proposition 2.15, the random variable  $(z_d - 1)\zeta_P(z)$  has operator stable distribution on  $\mathbb{C} \equiv \mathbb{R}^2$  with exponent matrix  $B$ ; see [32] for the definition of the exponent matrix. Moreover, this distribution is full in  $\mathbb{C} \equiv \mathbb{R}^2$  by Proposition 8.7. The spectrum of  $B$  is  $\operatorname{spec} B = \{z_1, \bar{z}_1\}$ . It is known that a full operator stable law has finite moments of all orders  $p \in (0, 1/\Lambda)$ , where  $\Lambda := \max\{\operatorname{Re} \lambda : \lambda \in \operatorname{spec} B\}$ ; see Theorem 3 in [22]. Also, it is known that the moments of order  $p > 1/\Lambda$  are infinite provided that  $\Lambda > \frac{1}{2}$ ; see Theorem 4 in [22]. In our case, we have  $\Lambda = \operatorname{Re} z_1 > \frac{1}{2}$ .  $\square$

## 9. THE FIRST LEVEL OF THE GREM

In this section, we collect some results on the *first level* of the GREM. These results will be used to obtain the fluctuations of  $\mathcal{Z}_n(\beta)$  in the Poissonian case  $|\sigma| > \frac{\sigma_1}{2}$ . (Though, let us stress that we do *not* assume this condition to hold throughout this section.

**9.1. Convergence to the Poisson process.** Recall that the first level of the GREM is labeled by the i.i.d. real standard Gaussian random variables  $\{\xi_k : 1 \leq k \leq N_{n,1}\}$ . It turns out that if the inverse temperature  $\beta \in \mathbb{C}$  is such that  $|\sigma| > \frac{\sigma_1}{2}$ , the main contribution of the first level to the partition function  $\mathcal{Z}_n(\beta)$  comes from the *extremal order statistics* among the  $\xi_k$ 's. (Upper order statistics for  $\sigma > 0$  and lower order statistics for  $\sigma < 0$ ). It is well-known from the standard extreme-value theory [28, Theorem 1.5.3] that the appropriately normalized upper order statistics of the  $\xi_k$ 's converge, as  $n \rightarrow \infty$ , to the Poisson point process with intensity  $e^{-u}du$  on  $\mathbb{R}$ . Namely, weakly on  $\mathcal{N}(\mathbb{R})$  it holds that

$$(9.1) \quad \sum_{k=1}^{N_{n,1}} \delta(u_{n,1}(\xi_k - u_{n,1})) \xrightarrow[n \rightarrow \infty]{w} \operatorname{PPP}(e^{-u}du).$$



Here, the normalizing sequence  $u_{n,1}$  is as in (2.22) and (2.23). For our purposes, it will be convenient to introduce a transformation of the energies at the first level which, as we will show in Lemma 9.2, maps the upper order statistics of the  $\xi_k$ 's to approximately a *homogeneous* Poisson point process on  $(0, \infty)$ . This transformation will be frequently used in our proofs. Define random variables  $\{P_{n,k} : n \in \mathbb{N}, 1 \leq k \leq N_{n,1}\}$  by

$$(9.2) \quad P_{n,k} = e^{-\sigma_1 \sqrt{na_1}} (\xi_k - u_{n,1}).$$

Note that, for every  $n \in \mathbb{N}$ , the random variables  $\{P_{n,k} : 1 \leq k \leq N_{n,1}\}$  are i.i.d.

**Lemma 9.1.** *For every  $z \in \mathbb{C}$  and  $0 < A < B$ ,*

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-z} \mathbb{1}_{A \leq P_{n,k} \leq B}] = \int_A^B y^{-z} dy.$$

*If, additionally,  $\operatorname{Re} z < 1$ , then the formula continues to hold even for  $0 \leq A < B$ .*

*Proof.* Introduce the real variables  $x$  and  $y$  such that  $y(x) = e^{-\sigma_1 \sqrt{na_1}} (x - u_{n,1})$  and, consequently,  $x(y) = u_{n,1} - \frac{\log y}{\sigma_1 \sqrt{na_1}}$ . The transformation  $x \mapsto y$  is a monotone decreasing bijection between  $\mathbb{R}$  and  $(0, \infty)$ . Recalling that  $P_{n,k} = y(\xi_k)$ , where  $\xi_k \sim N_{\mathbb{R}}(0, 1)$ , we obtain

$$\begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-z} \mathbb{1}_{A \leq P_{n,k} \leq B}] &= N_{n,1} \frac{1}{\sqrt{2\pi}} \int_{y^{-1}(B)}^{y^{-1}(A)} (y(x))^{-z} e^{-\frac{1}{2}x^2} dx \\ &= N_{n,1} \frac{1}{\sqrt{2\pi}} \int_A^B y^{-z} e^{-\frac{1}{2}(u_{n,1} - \frac{\log y}{\sigma_1 \sqrt{na_1}})^2} \frac{dy}{y \sigma_1 \sqrt{na_1}} \\ &= N_{n,1} \frac{e^{-\frac{1}{2}u_{n,1}^2}}{\sqrt{2\pi} \sigma_1 \sqrt{na_1}} \int_A^B y^{-z-1} e^{u_{n,1} \frac{\log y}{\sigma_1 \sqrt{na_1}}} e^{-\frac{1}{2} \frac{(\log y)^2}{\sigma_1^2 na_1}} dy. \end{aligned}$$

Using the fact that  $u_{n,1} \sim \sigma_1 \sqrt{na_1}$ , see (2.23), the relation between  $u_{n,1}$  and  $N_{n,1}$ , see (2.22), and the dominated convergence theorem (note that the integrand can be estimated by  $y^{-\operatorname{Re} z \pm \varepsilon}$ , for sufficiently large  $n$ ), we obtain that the right-hand side converges to  $\int_A^B y^{-z} dy$ .  $\square$

The next lemma is just a reformulation of (9.1).

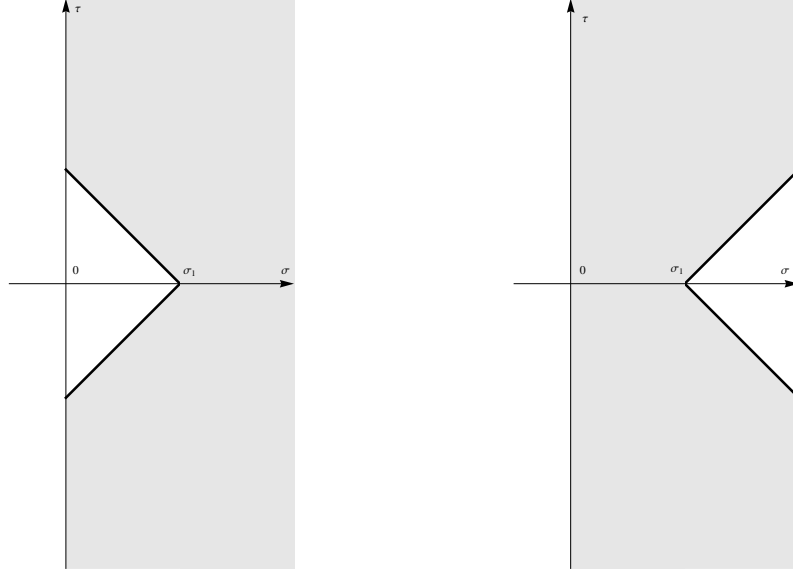
**Lemma 9.2.** *Let  $\sum_{k=1}^{\infty} \delta(P_k)$  be a unit intensity Poisson point process on  $(0, \infty)$ . The following convergence of point processes holds weakly on  $\mathcal{N}([0, \infty))$ :*

$$\sum_{k=1}^{N_{n,1}} \delta(P_{n,k}) \xrightarrow[n \rightarrow \infty]{w} \sum_{k=1}^{\infty} \delta(P_k).$$

*Proof.* Let  $\mu_n$  be the probability distribution of  $P_{n,k}$ . Then, Lemma 9.1 with  $z = 0$  implies that the measure  $N_{n,1} \mu_n$  converges vaguely to the Lebesgue measure on  $[0, \infty)$ . Since the variables  $\{P_{n,k} : 1 \leq k \leq N_{n,1}\}$  are i.i.d., this yields the desired weak convergence by [37, Proposition 3.21].  $\square$

**9.2. Asymptotics for the truncated moments of  $P_{n,k}$ .** In this section, we compute the limits

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] \quad \text{and} \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} < 1}].$$



Domain in which Lemma 9.3 is valid.

Domain in which Lemma 9.4 is valid.

FIGURE 10. Asymptotics for the truncated moments of  $P_{n,k}$ 

As we have shown in Lemma 9.1 (with  $z = 0$ ), the probability distribution  $\mu_n$  of  $P_{n,k}$ , multiplied by  $N_{n,1}$ , converges vaguely to the Lebesgue measure on  $(0, \infty)$ . One could therefore try to proceed as follows:

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] = \lim_{n \rightarrow \infty} N_{n,1} \int_1^\infty p^{-\frac{\beta}{\sigma_1}} \mu_n(dp) = \int_1^\infty p^{-\frac{\beta}{\sigma_1}} dp = \frac{\sigma_1}{\beta - \sigma_1}.$$

This approach works in the half-plane  $\sigma > \sigma_1$  since under this condition the integral  $\int_1^\infty p^{-\beta/\sigma_1} dp$  is convergent. However, as we will show in the next lemma, the above formula is valid in a domain which is *strictly larger* than the half-plane  $\sigma > \sigma_1$ . This fact is crucial because, as we will see later, it is responsible for the beak shaped form of the boundary between the phases  $G_k$  and  $E_k$ .

**Lemma 9.3.** *Let  $K$  be a compact subset of  $\{\beta \in \mathbb{C} : \sigma > 0, \sigma + |\tau| > \sigma_1\}$ ; see Figure 10, left. Then, uniformly in  $\beta \in K$  we have*

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] = \frac{\sigma_1}{\beta - \sigma_1}.$$

*Proof.* Let  $\xi \sim N_{\mathbb{R}}(0, 1)$  be real standard normal variable. By the definition of  $P_{n,k}$ , see (9.2), and by Lemma 3.8, Part 2, we have

$$\begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] &= N_{n,1} \mathbb{E}[e^{\beta \sqrt{na_1}(\xi - u_{n,1})} \mathbb{1}_{\xi < u_{n,1}}] \\ &= N_{n,1} e^{-\beta \sqrt{na_1} u_{n,1}} e^{\frac{1}{2} \beta^2 na_1} \Phi(u_{n,1} - \beta \sqrt{na_1}). \end{aligned}$$

In the first step, we have used that  $\sigma > 0$ . Since  $u_{n,1} \sim \sigma_1 \sqrt{na_1}$  by (2.23), we have

$$\operatorname{Re}(u_{n,1} - \beta \sqrt{na_1}) \sim (\sigma_1 - \sigma) \sqrt{na_1}, \quad \operatorname{Im}(u_{n,1} - \beta \sqrt{na_1}) \sim -\tau \sqrt{na_1}.$$

It follows from the assumption that  $K$  is a compact subset of  $\{\sigma + |\tau| > \sigma_1\}$  that we can find  $\varepsilon > 0$  such that, for all  $\beta \in K$  and for all sufficiently large  $n \in \mathbb{N}$ ,

$$u_{n,1} - \beta \sqrt{na_1} \in \left\{ z \in \mathbb{C} : |\arg z| > \frac{\pi}{4} + \varepsilon \right\}.$$

Hence, we can apply Lemma 3.10 to obtain that

$$\begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] &\sim -N_{n,1} e^{-\beta \sqrt{na_1} u_{n,1}} e^{\frac{1}{2} \beta^2 na_1} \frac{e^{-\frac{1}{2} (u_{n,1} - \beta \sqrt{na_1})^2}}{\sqrt{2\pi} (u_{n,1} - \beta \sqrt{na_1})} \\ &= \frac{N_{n,1}}{\sqrt{2\pi} u_{n,1} e^{\frac{1}{2} u_{n,1}^2}} \cdot \frac{u_{n,1}}{\beta \sqrt{na_1} - u_{n,1}} \end{aligned}$$

The right-hand side converges to  $\frac{\sigma_1}{\beta - \sigma_1}$  by (2.22) and (2.23).  $\square$

**Lemma 9.4.** *Let  $K$  be a compact subset of  $\{\beta \in \mathbb{C} : \sigma > 0, \sigma - |\tau| < \sigma_1\}$ ; see Figure 10, right. Then, uniformly in  $\beta \in K$ ,*

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} < 1}] = -\frac{\sigma_1}{\beta - \sigma_1}.$$

*Proof.* The proof is similar to the proof of Lemma 9.3. By the definition of  $P_{n,k}$ , see (9.2), and by Lemma 3.8, Part 1, we have

$$\begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} < 1}] &= N_{n,1} \mathbb{E}[e^{\beta \sqrt{na_1} (\xi - u_{n,1})} \mathbb{1}_{\xi > u_{n,1}}] \\ &= N_{n,1} e^{-\beta \sqrt{na_1} u_{n,1}} e^{\frac{1}{2} \beta^2 na_1} \bar{\Phi}(u_{n,1} - \beta \sqrt{na_1}) \\ &= N_{n,1} e^{-\beta \sqrt{na_1} u_{n,1}} e^{\frac{1}{2} \beta^2 na_1} \Phi(\beta \sqrt{na_1} - u_{n,1}), \end{aligned}$$

where in the first equality we used that  $\sigma > 0$  and in the last step we used that  $\bar{\Phi}(z) = \Phi(-z)$ . Since  $u_{n,1} \sim \sigma_1 \sqrt{na_1}$  by (2.23), we have

$$\operatorname{Re}(\beta \sqrt{na_1} - u_{n,1}) \sim (\sigma - \sigma_1) \sqrt{na_1}, \quad \operatorname{Im}(\beta \sqrt{na_1} - u_{n,1}) \sim \tau \sqrt{na_1}.$$

It follows from the assumption that  $K$  is a compact subset of  $\{\sigma - |\tau| < \sigma_1\}$  that there is  $\varepsilon > 0$  such that for all  $\beta \in K$  and for all sufficiently large  $n \in \mathbb{N}$ ,

$$\beta \sqrt{na_1} - u_{n,1} \in \left\{ z \in \mathbb{C} : |\arg z| > \frac{\pi}{4} + \varepsilon \right\}.$$

Hence, we can apply Lemma 3.10 to obtain that

$$\begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} < 1}] &\sim -N_{n,1} e^{-\beta \sqrt{na_1} u_{n,1}} e^{\frac{1}{2} \beta^2 na_1} \frac{e^{-\frac{1}{2} (\beta \sqrt{na_1} - u_{n,1})^2}}{\sqrt{2\pi} (\beta \sqrt{na_1} - u_{n,1})} \\ &= -\frac{N_{n,1}}{\sqrt{2\pi} u_{n,1} e^{\frac{1}{2} u_{n,1}^2}} \cdot \frac{u_{n,1}}{\beta \sqrt{na_1} - u_{n,1}} \end{aligned}$$

The right-hand side converges to  $-\frac{\sigma_1}{\beta - \sigma_1}$  by (2.22) and (2.23).  $\square$

**9.3. Estimates for the truncated moments of  $P_{n,k}$ .** In the next lemmata, we prove some estimates on the truncated moments of  $P_{n,k}$ .

**Lemma 9.5.** *Let  $K$  be a compact subset of  $\{\beta \in \mathbb{C}: 0 \leq \sigma < \sigma_1\}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$N_{n,1} \mathbb{E} |P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1}| \leq C.$$

*Proof.* Let  $\xi \sim N_{\mathbb{R}}(0, 1)$ . By definition of  $P_{n,k}$ , see (9.2),

$$N_{n,1} \mathbb{E} |P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1}| = N_{n,1} \mathbb{E} [e^{\sigma \sqrt{na_1} (\xi - u_{n,1})} \mathbb{1}_{\xi \geq u_{n,1}}].$$

We are going to apply Lemma 3.9, Part 1, with  $w = \sigma \sqrt{na_1}$  and  $a = u_{n,1} \sim \sigma_1 \sqrt{na_1}$ . Since  $K$  is a compact subset of  $\{\sigma < \sigma_1\}$ , there exist  $n_0 \in \mathbb{N}$ ,  $\varepsilon > 0$  such that  $a > w$  and moreover  $a - w > \varepsilon u_{n,1}$  for all  $n > n_0$ ,  $\beta \in K$ . By Lemma 3.9, Part 1, for all  $n > n_0$  and  $\beta \in K$ ,

$$N_{n,1} \mathbb{E} [e^{\sigma \sqrt{na_1} (\xi - u_{n,1})} \mathbb{1}_{\xi \geq u_{n,1}}] \leq \frac{CN_{n,1} e^{-\frac{1}{2} u_{n,1}^2}}{u_{n,1} - \sigma \sqrt{na_1}} \leq CN_{n,1} u_{n,1}^{-1} e^{-\frac{1}{2} u_{n,1}^2}.$$

By (2.22), the right-hand side is bounded by  $C$ . If necessary, we can enlarge  $C$  so that the estimate holds for all  $n \in \mathbb{N}$ .  $\square$

**Lemma 9.6.** *Let  $K$  be a compact subset of  $\{\beta \in \mathbb{C}: \sigma > \sigma_1\}$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$ ,  $T \geq 1$  and all sufficiently large  $n \in \mathbb{N}$ ,*

$$N_{n,1} |\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T})| \leq N_{n,1} \mathbb{E} |P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}| < CT^{-\varepsilon}.$$

*Proof.* Let  $\xi \sim N_{\mathbb{R}}(0, 1)$ . By definition of  $P_{n,k}$ , see (9.2),

$$N_{n,1} \mathbb{E} |P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}| = N_{n,1} \mathbb{E} \left[ e^{\sigma \sqrt{na_1} (\xi - u_{n,1})} \mathbb{1}_{\xi < u_{n,1} - \frac{\log T}{\sigma_1 \sqrt{na_1}}} \right].$$

We are going to apply Lemma 3.9, Part 2, with  $w = \sigma \sqrt{na_1}$  and  $a = u_{n,1} - \frac{\log T}{\sigma_1 \sqrt{na_1}}$ . We have  $a \leq u_{n,1} \sim \sigma_1 \sqrt{na_1}$  and hence, for sufficiently large  $n$ ,  $a < w$  and moreover,  $w - a > \eta u_{n,1}$  for some sufficiently small constant  $\eta > 0$ . Therefore, by Lemma 3.9,

$$\begin{aligned} N_{n,1} \mathbb{E} |P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}| &\leq \frac{CN_{n,1} e^{-\frac{\sigma}{\sigma_1} \log T}}{\sigma \sqrt{na_1} - u_{n,1} + \frac{\log T}{\sigma_1 \sqrt{na_1}}} \exp \left\{ -\frac{1}{2} \left( u_{n,1} - \frac{\log T}{\sigma_1 \sqrt{na_1}} \right)^2 \right\} \\ &\leq CT^{\left( \frac{u_{n,1}}{\sigma_1 \sqrt{na_1}} - \frac{\sigma}{\sigma_1} \right)} \frac{N_{n,1}}{u_{n,1}} \exp \left\{ -\frac{1}{2} u_{n,1}^2 \right\}. \end{aligned}$$

Since  $u_{n,1} \sim \sigma_1 \sqrt{na_1}$  and  $K$  is a compact subset of  $\{\sigma > \sigma_1\}$ , we can find  $n_0 \in \mathbb{N}$  and  $\varepsilon > 0$  such that for all  $\beta \in K$  and  $n > n_0$ ,

$$\frac{u_{n,1}}{\sigma_1 \sqrt{na_1}} - \frac{\sigma}{\sigma_1} < -\varepsilon.$$

Recalling (2.22) we obtain the required estimate.  $\square$

**Lemma 9.7.** *Let  $K$  be a compact subset of  $\{\beta \in \mathbb{C}: \sigma > 0, \sigma + |\tau| > \sigma_1\}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K$ ,  $T \geq 1$  and all sufficiently large  $n \in \mathbb{N}$ ,*

$$N_{n,1} |\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T})| < CT.$$

*Proof.* By the definition of  $P_{n,k}$ , see (9.2), and Lemma 3.8, Part 2, we have

$$\begin{aligned} N_{n,1} \mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}) &= N_{n,1} \mathbb{E}[e^{\beta\sqrt{na_1}(\xi - u_{n,1})} \mathbb{1}_{\xi < a_n(T)}] \\ &= N_{n,1} e^{-\beta\sqrt{na_1}u_{n,1}} e^{\frac{1}{2}\beta^2 na_1} \Phi(a_n(T) - \beta\sqrt{na_1}), \end{aligned}$$

where  $a_n(T) = u_{n,1} - \frac{\log T}{\sigma_1\sqrt{na_1}}$ . Since  $u_{n,1} \sim \sigma_1\sqrt{na_1}$  by (2.23), we have

$$\begin{aligned} \operatorname{Re}(a_n(T) - \beta\sqrt{na_1}) &< u_{n,1} - \sigma\sqrt{na_1} = (\sigma_1 - \sigma)\sqrt{na_1} + o(\sqrt{n}), \\ \operatorname{Im}(a_n(T) - \beta\sqrt{na_1}) &= -\tau\sqrt{na_1} + o(\sqrt{n}), \end{aligned}$$

where the  $o$ -term is uniform in  $T$ . It follows from the assumption that  $K$  is a compact subset of  $\{\sigma + |\tau| > \sigma_1\}$  that there is  $\varepsilon > 0$  such that for all  $n \in \mathbb{N}$ ,  $T \geq 1$ ,  $\beta \in K$ ,

$$a_n(T) - \beta\sqrt{na_1} \in \left\{ z \in \mathbb{C} : |\arg z| > \frac{\pi}{4} + \varepsilon \right\}.$$

Hence, we can use Lemma 3.10 to obtain that uniformly in  $\beta \in K$  and  $T \geq 1$ ,

$$\begin{aligned} N_{n,1} \mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}) &\sim C N_{n,1} e^{-\beta\sqrt{na_1}u_{n,1}} e^{\frac{1}{2}\beta^2 na_1} \cdot \frac{e^{-\frac{1}{2}(a_n(T) - \beta\sqrt{na_1})^2}}{a_n(T) - \beta\sqrt{na_1}} \\ &= \frac{C N_{n,1} e^{-\frac{1}{2}u_{n,1}^2}}{a_n(T) - \beta\sqrt{na_1}} \cdot T^{\frac{u_{n,1} - \beta\sqrt{na_1}}{\sigma_1\sqrt{na_1}}} \cdot e^{-\frac{1}{2}\frac{(\log T)^2}{\sigma_1^2 na_1}}. \end{aligned}$$

We are going to show that there is a sufficiently small  $\delta > 0$  such that, for all  $T \geq 1$ ,  $\beta \in K$  and all sufficiently large  $n$ ,

$$(9.3) \quad \frac{u_{n,1} - \sigma\sqrt{na_1}}{\sigma_1\sqrt{na_1}} \leq 1,$$

$$(9.4) \quad |a_n(T) - \beta\sqrt{na_1}| > \delta u_{n,1}.$$

After (9.3) and (9.4) have been established, the proof of the lemma can be completed by recalling (2.22).

*Proof of (9.3).* The left-hand side in (9.3) converges to  $1 - \frac{\sigma}{\sigma_1} < 1$ , since  $K$  is assumed to be a compact subset of  $\{\sigma > 0\}$ .

*Proof of (9.4).* We can find a sufficiently small  $\varepsilon > 0$  such that  $K$  is contained in the union of the sets  $\{|\tau| > \varepsilon\}$  and  $\{\sigma > \sigma_1 + \varepsilon\}$ .

CASE 1:  $|\tau| > \varepsilon$ . For sufficiently large  $n$ , we have

$$|a_n(T) - \beta\sqrt{na_1}| \geq |\operatorname{Im}(a_n(T) - \beta\sqrt{na_1})| = |\tau|\sqrt{na_1} > \delta u_{n,1}.$$

CASE 2:  $\sigma > \sigma_1 + \varepsilon$ . For large enough  $n$  and all  $T \geq 1$ , we have

$$\operatorname{Re}(a_n(T)) \leq u_{n,1} < \left(\sigma_1 + \frac{\varepsilon}{2}\right)\sqrt{na_1} < \left(\sigma - \frac{\varepsilon}{2}\right)\sqrt{na_1}.$$

Hence,

$$|a_n(T) - \beta\sqrt{na_1}| \geq |\operatorname{Re}(a_n(T) - \beta\sqrt{na_1})| > \frac{\varepsilon}{2}\sqrt{na_1} > \delta u_{n,1}.$$

This completes the proof of (9.4).  $\square$

**9.4. Adjoining the remaining levels.** The next lemma will be used when we adjoin a new Poissonian level to a GREM with  $d - 1$  levels. In this lemma, one should think of  $P_{n,k}$  as the contributions of the first level of the GREM and of  $\mathbf{Z}_{n,k}$  as the contributions of the remaining  $d - 1$  levels.

**Lemma 9.8.** *Let  $P_{n,k}$  be as above, see (9.2), and independently, for every  $n \in \mathbb{N}$ , let  $\{\mathbf{Z}_{n,k} : 1 \leq k \leq N_{n,1}\}$  be i.i.d.  $\mathbb{C}^r$ -valued random vectors with  $\mathbf{Z}_{n,k} = \{\mathbf{Z}_{n,k}(i)\}_{i=1}^r$ . Assume that  $\mathbf{Z}_{n,k}$  converges in distribution to some random vector  $\mathbf{Z} = \{\mathbf{Z}(i)\}_{i=1}^r$ , as  $n \rightarrow \infty$ . Let also  $c_{n,1}, \dots, c_{n,r} \in \mathbb{C}$  be sequences such that  $c_i := \lim_{n \rightarrow \infty} c_{n,i} \in \mathbb{C}$  exists, for all  $1 \leq i \leq r$ . Then, for every  $T > 0$ , we have the following weak convergence of random vectors in  $\mathbb{C}^r$ :*

$$(9.5) \quad \left\{ \sum_{k=1}^{N_{n,1}} P_{n,k}^{-c_{n,i}} \mathbb{1}_{P_{n,k} \leq T} \mathbf{Z}_{n,k}(i) \right\}_{i=1}^r \xrightarrow[n \rightarrow \infty]{d} \left\{ \sum_{k=1}^{\infty} P_k^{-c_i} \mathbb{1}_{P_k \leq T} \mathbf{Z}_k(i) \right\}_{i=1}^r,$$

where  $\sum_{k=1}^{\infty} \delta(P_k)$  is a unit intensity Poisson point process on  $[0, \infty)$  and, independently,  $\mathbf{Z}_1, \mathbf{Z}_2, \dots$  are i.i.d. copies of  $\mathbf{Z}$ .

*Proof.* STEP 1. Denote the vector on the left-hand side of (9.5) by  $\mathbf{S}_n = \{\mathbf{S}_n(i)\}_{i=1}^r$  and the vector on the right-hand side of (9.5) by  $\mathbf{S} = \{\mathbf{S}(i)\}_{i=1}^r$ . We have to show that, for every continuous bounded function  $f : \mathbb{C}^r \rightarrow \mathbb{R}$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E}f(\mathbf{S}_n) = \mathbb{E}f(\mathbf{S}).$$

Let  $K_n$  be the number of points  $P_{n,k}$ ,  $1 \leq k \leq N_{n,1}$ , which satisfy  $P_{n,k} \leq T$ . Similarly, let  $K$  be the number of points  $P_k$ ,  $k \in \mathbb{N}$ , which satisfy  $P_k \leq T$ . By the total expectation formula, we need to show that

$$(9.6) \quad \lim_{n \rightarrow \infty} \sum_{l=0}^{\infty} \mathbb{E}[f(\mathbf{S}_n) | K_n = l] \mathbb{P}[K_n = l] = \sum_{l=0}^{\infty} \mathbb{E}[f(\mathbf{S}) | K = l] \mathbb{P}[K = l].$$

The proof of (9.6) follows from Steps 2, 3, 4 below.

STEP 2. By Lemma 9.1 (with  $z = 0$ ) and the Poisson limit theorem, for every  $l \in \mathbb{N}_0$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[K_n = l] = \mathbb{P}[K = l].$$

STEP 3. We show that for every  $l \in \mathbb{N}_0$ ,

$$(9.7) \quad \lim_{n \rightarrow \infty} \mathbb{E}[f(\mathbf{S}_n) | K_n = l] = \mathbb{E}[f(\mathbf{S}) | K = l].$$

Let  $\mu_n$  be the distribution of  $P_{n,k}$ . Conditionally on  $K_n = l$ , those random variables  $P_{n,k}$  that satisfy  $P_{n,k} \leq T$  have the same joint distribution as the i.i.d. random variables  $(Q_{n,1}, \dots, Q_{n,l})$  distributed on  $[0, T]$  according to the measure  $\mu_n(\cdot)/\mu_n([0, T])$ . This distribution converges weakly to the uniform distribution on  $[0, T]$  by Lemma 9.1 (with  $z = 0$ ) and hence,  $(Q_{n,1}, \dots, Q_{n,l})$  converges in distribution to i.i.d. random variables  $(Q_1, \dots, Q_l)$  distributed uniformly on  $[0, T]$ . We have

$$\mathbf{S}_n | \{K_n = l\} \stackrel{d}{=} \left\{ \sum_{j=1}^l Q_{n,j}^{-c_{n,i}} \mathbf{Z}_{n,j}(i) \right\}_{i=1}^r \xrightarrow[n \rightarrow \infty]{d} \left\{ \sum_{j=1}^l Q_j^{-c_i} \mathbf{Z}_j(i) \right\}_{i=1}^r \stackrel{d}{=} \mathbf{S} | \{K = l\},$$

This proves (9.7).

STEP 4. To complete the proof of (9.6), we need to show that

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{l=k}^{\infty} \mathbb{E}[f(\mathbf{S}_n) | K_n = l] \mathbb{P}[K_n = l] = 0.$$

However, this follows from the estimate

$$\limsup_{n \rightarrow \infty} \sum_{l=k}^{\infty} \mathbb{E}[f(\mathbf{S}_n) | K_n = l] \mathbb{P}[K_n = l] \leq C \lim_{n \rightarrow \infty} \mathbb{P}[K_n \geq k] = \mathbb{P}[K \geq k],$$

where we used the boundedness of the function  $f$  and Step 2.  $\square$

## 10. MOMENT ESTIMATES IN PHASES WITHOUT FLUCTUATION LEVELS

**10.1. Introduction and notation.** In this section, we obtain estimates for the moments of  $\mathcal{Z}_n(\beta)$  and some related processes in phases of the form  $G^{d_1} E^{d-d_1}$ , where  $0 \leq d_1 \leq d$ . The main results of this section, Proposition 10.2 and Lemma 10.8, will be a crucial ingredient in the proofs of functional limit theorems in Section 11. Some of our most important moment estimates will be valid in the domain

$$\mathcal{O} = \left( E_2 \cup \left( \bigcup_{d_1=2}^d G^{d_1} E^{d-d_1} \right) \right) \cap \{\sigma > 0\}.$$

Note that the set  $\mathcal{O}$  is open. It *does* include the beak shaped boundary between  $E_1 = E^d$  and  $G^1 E^{d-1}$  but it *does not* include the boundaries between  $G^{d_1-1} E^{d-d_1+1}$  and  $G^{d_1} E^{d-d_1}$  for  $2 \leq d_1 \leq d$ .

To state our results, we need to define  $S_n(\beta)$  and  $S_n^\circ(\beta)$ , two normalized versions of the random partition function  $\mathcal{Z}_n(\beta)$ . It turns out that  $S_n^\circ(\beta)$  is the “correct” normalization the sense that  $S_n^\circ(\beta)$  has non-trivial limiting fluctuations in  $\mathcal{O} \cap \{\sigma > \frac{\sigma_1}{2}\}$ ; see for example Theorem 11.1 below. First, we define a normalizing sequence

$$(10.1) \quad \tilde{c}_n(\beta) = c_{n,2}(\beta) + \dots + c_{n,d}(\beta),$$

where, for  $2 \leq k \leq d$  and  $\beta \in \mathcal{O}$ ,

$$(10.2) \quad c_{n,k}(\beta) = \begin{cases} \beta \sqrt{na_k} u_{n,k}, & \text{if } \beta \in G_k, \\ \log N_{n,k} + \frac{1}{2} a_k \beta^2 n, & \text{if } \beta \in E_k. \end{cases}$$

Think of  $e^{\tilde{c}_n(\beta)}$  as of the sequence needed to normalize the levels  $2, \dots, d$ . Let  $\{S_n(\beta) : \beta \in \mathcal{O}\}$  and  $\{S_n^\circ(\beta) : \beta \in \mathcal{O}\}$  be random analytic functions defined by

$$(10.3) \quad S_n(\beta) = \frac{\mathcal{Z}_n(\beta)}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}},$$

$$(10.4) \quad S_n^\circ(\beta) = \frac{\mathcal{Z}_n(\beta) - e^{\tilde{c}_n(\beta)} N_{n,1} \mathbb{E}[e^{\beta \sqrt{na_1} \xi} \mathbb{1}_{\xi < u_{n,1}}]}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}}.$$

Note that in  $S_n(\beta)$  the  $k$ -th level, for  $2 \leq k \leq d$ , is normalized by the expectation if  $\beta \in E_k$  or by the order of the maximal energy on this level if  $\beta \in G_k$ . The first level is always normalized by the order of the maximal energy, even in the case  $\beta \in E_1 \cap \{\sigma > \frac{\sigma_1}{2}\}$  (where normalization by expectation may seem more natural at a first sight). Note also that  $S_n^\circ(\beta)$  differs from  $S_n(\beta)$  by an additional additive normalization. In the sequel, we agree to mark by  $^\circ$  random variables normalized by some sort of truncated expectation.

**10.2. Second moment estimate in  $\{|\beta| < \frac{\sigma_1}{\sqrt{2}}\}$ .** We start with a simple second moment estimate for  $\mathcal{Z}_n(\beta)$ .

**Proposition 10.1.** *Let  $K$  be a compact subset of the disk  $\{|\beta| < \frac{\sigma_1}{\sqrt{2}}\}$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^2 \leq C e^{-\varepsilon n}.$$

*Proof.* Using Propositions 2.6, 2.5 and then (1.1) and (1.8), we obtain that uniformly in  $K$ ,

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^2 = \frac{\text{Var } \mathcal{Z}_n(\beta)}{|\mathbb{E}\mathcal{Z}_n(\beta)|^2} \sim N_{n,1}^{-1} e^{|\beta|^2 a_1 n} = e^{-(\frac{1}{2}\sigma_1^2 - |\beta|^2) a_1 n + o(1)}.$$

Since  $\frac{1}{2}\sigma_1^2 - |\beta|^2$  admits a strictly positive uniform lower bound on  $K$ , we can estimate the right-hand side by  $e^{-\varepsilon n}$ , for sufficiently large  $n$ . Choosing the constant  $C$  large enough, we can achieve that the estimate holds for all  $n \in \mathbb{N}$ .  $\square$

**10.3. The main estimate and its corollaries.** Unfortunately, the second moment estimate of Proposition 10.1 is valid in a very small domain only. In order to obtain estimates for larger domains, we need to replace the second moment by the moment of order  $p \in (0, 2)$ . The main result of this section can be stated as follows.

**Proposition 10.2.** *Fix  $p \in (0, 2)$ .*

- (1) *Let  $K$  be a compact subset of  $E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \frac{\sigma_1}{p}\}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$(10.5) \quad \mathbb{E}|S_n^\circ(\beta)|^p < C.$$

- (2) *Let  $K$  be a compact subset of  $G^{d_1} E^{d-d_1} \cap \{0 \leq \sigma < \frac{\sigma_1}{p}\}$ , where  $1 \leq d_1 \leq d$ . Then, there is a constant  $C = C(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$(10.6) \quad \mathbb{E}|S_n(\beta)|^p \leq C, \quad \mathbb{E}|S_n^\circ(\beta)|^p \leq C,$$

From the first part of Proposition 10.2, we can draw the following corollaries on the moments of  $\mathcal{Z}_n(\beta)$  in  $E_1$ .

**Corollary 10.3.** *Fix  $p \in (0, 2)$ . Let  $K$  be a compact subset of  $E_1 \cap \{|\sigma| < \frac{\sigma_1}{p}\}$ . Then, there exist  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$(10.7) \quad \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^p < C e^{-\varepsilon n}.$$

**Corollary 10.4.** *Fix  $p \in (0, 2)$ . Let  $K$  be a compact subset of  $E_1 \cap \{|\sigma| < \frac{\sigma_1}{p}\}$ . Then, there exists  $C = C(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$(10.8) \quad \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} \right|^p < C.$$

Before turning to the proof of Proposition 10.2, we show how to deduce Corollaries 10.3 and 10.4 from Proposition 10.2.



*Proof of Corollary 10.3 given Proposition 10.2.* By Proposition 10.1, together with Lyapunov's inequality (3.1), the required estimate (10.7) holds in any compact subset of the disk  $\{|\beta| < \frac{\sigma_1}{\sqrt{2}}\}$ .

Therefore, in the rest of the proof we may assume that  $K$  is a compact subset of  $E_1 \cap \{\frac{\sigma_1}{2} < |\sigma| < \frac{\sigma_1}{p}\}$ . By symmetry, see (1.10), we can also assume that  $K \subset \{\sigma \geq 0\}$ . Expressing  $\mathcal{Z}_n(\beta)$  in terms of  $S_n^\circ(\beta)$ , see (10.4), we obtain

$$(10.9) \quad \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^p = \mathbb{E} \left| \frac{S_n^\circ(\beta)e^{\beta\sqrt{na_1}u_{n,1}}}{e^{\frac{1}{2}\beta^2na_1}N_{n,1}} + \frac{\mathbb{E}[e^{\beta\sqrt{na_1}\xi}\mathbb{1}_{\xi < u_{n,1}}]}{e^{\frac{1}{2}\beta^2na_1}} - 1 \right|^p.$$

Applying Lemma 3.8, we obtain

$$\frac{\mathbb{E}[e^{\beta\sqrt{na_1}\xi}\mathbb{1}_{\xi < u_{n,1}}]}{e^{\frac{1}{2}\beta^2na_1}} - 1 = -\frac{\mathbb{E}[e^{\beta\sqrt{na_1}\xi}\mathbb{1}_{\xi > u_{n,1}}]}{e^{\frac{1}{2}\beta^2na_1}} = -\bar{\Phi}(u_{n,1} - \beta\sqrt{na_1}).$$

By Jensen's inequality (3.2) applied to (10.9),

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^p \leq C\mathbb{E}|S_n^\circ(\beta)|^p \left( \frac{e^{\sigma\sqrt{na_1}u_{n,1}}}{e^{\frac{1}{2}(\sigma^2 - \tau^2)na_1}N_{n,1}} \right)^p + C(\bar{\Phi}(u_{n,1} - \beta\sqrt{na_1}))^p.$$

To complete the proof, we need to estimate the terms on the right-hand side. This will be done in 3 steps.

STEP 1. By Proposition 10.2, Part 1, there is a constant  $C = C(K) > 0$  such that  $\mathbb{E}|S_n^\circ(\beta)|^p < C$  for all  $\beta \in K$  and all  $n \in \mathbb{N}$ .

STEP 2. Recall that  $u_{n,1} \sim \sigma_1\sqrt{na_1}$  and  $N_{n,1} = e^{\frac{1}{2}\sigma_1^2na_1 + o(n)}$ ; see (2.23) and (1.1). It follows that

$$\frac{e^{\sigma\sqrt{na_1}u_{n,1}}}{e^{\frac{1}{2}(\sigma^2 - \tau^2)na_1}N_{n,1}} = e^{-\frac{1}{2}na_1((\sigma_1 - \sigma)^2 - \tau^2) + o(n)} < e^{-\varepsilon n},$$

for suitable  $\varepsilon > 0$  and all sufficiently large  $n$ . Here, we have used the fact that  $(\sigma_1 - \sigma)^2 - \tau^2$  admits a strictly positive uniform lower bound on  $K$  since  $K$  is a compact subset of  $E_1$ .

STEP 3. Let  $z_n = u_{n,1} - \beta\sqrt{na_1}$ . Then,  $\operatorname{Re} z_n \sim (\sigma_1 - \sigma)\sqrt{na_1}$  and  $\operatorname{Im} z_n \sim -\tau\sqrt{na_1}$ . Since  $K$  is a compact subset of  $E_1$ , it follows that there is  $\delta = \delta(K) > 0$  such that  $|\arg z_n| < \frac{\pi}{4} - \delta < \frac{3\pi}{4} - \delta$  for all sufficiently large  $n$  and all  $\beta \in K$ . By Lemma 3.10 (second line of (3.6)), we have

$$|\bar{\Phi}(u_{n,1} - \beta\sqrt{na_1})| < e^{-\frac{1}{2}na_1((\sigma_1 - \sigma)^2 - \tau^2) + o(n)} < e^{-\varepsilon n},$$

for sufficiently large  $n$ , where the final estimate uses the same argumentation as in Step 2.

Combining the 3 steps, we get the required estimate (10.7), for sufficiently large  $n$ . By enlarging  $C$ , if necessary, we can achieve that it holds for all  $n$ .  $\square$

*Proof of Corollary 10.4 given Corollary 10.3.* By Jensen's inequality (3.2), we have

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} \right|^p \leq C\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right|^p + C.$$

The required estimate (10.8) follows from Corollary 10.3.  $\square$

**10.4. Proof of Proposition 10.2.** We use induction over  $d$ , the number of levels of the GREM. Assume that Proposition 10.2 (and hence, Corollaries 10.3 and 10.4) are valid for any GREM with  $d-1$  levels. Our aim is to prove that Proposition 10.2 holds for a GREM with  $d$  levels. The main idea is to consider the first level separately, and to apply the induction assumption to the remaining  $d-1$  levels. All variables which refer to these remaining levels will be marked by a tilde “ $\sim$ ”. For example, we define an index set

$$(10.10) \quad \tilde{\mathbb{S}}_n = \{\tilde{\varepsilon} = (\varepsilon_2, \dots, \varepsilon_d) \in \mathbb{N}^d : 1 \leq \varepsilon_2 \leq N_{2,n}, \dots, 1 \leq \varepsilon_d \leq N_{d,n}\}.$$

Define the random variables  $P_{n,k}$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the first level of the GREM) and the random analytic functions  $\{\tilde{Z}_{n,k}(\beta) : \beta \in \mathcal{O}\}$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the remaining  $d-1$  levels of the GREM) by

$$(10.11) \quad P_{n,k} = e^{-\sigma_1 \sqrt{n a_1} (\xi_k - u_{n,1})},$$

$$(10.12) \quad \tilde{Z}_{n,k}(\beta) = e^{-\tilde{c}_n(\beta)} \sum_{\tilde{\varepsilon} \in \tilde{\mathbb{S}}_n} e^{\beta \sqrt{n} (\sqrt{a_2} \xi_{k\varepsilon_2} + \dots + \sqrt{a_d} \xi_{k\varepsilon_2 \dots \varepsilon_d})}.$$

By the definition of the GREM, these random variables have the following properties, for every  $n \in \mathbb{N}$ :

- (1)  $\tilde{Z}_{n,k}(\beta)$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random processes.
- (2)  $P_{n,k}$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random variables.
- (3) These two collections are independent.

The properties of  $P_{n,k}$  have been studied in Section 9. It is useful to keep in mind that  $\{P_{n,k} : 1 \leq k \leq N_{n,1}\}$  is approximatively (for  $n \rightarrow \infty$ ) a homogeneous Poisson point process on  $(0, \infty)$ ; see Lemma 9.2. Note also that, for  $\beta \in E_2$ , we have  $\mathbb{E} \tilde{Z}_{n,k}(\beta) = 1$ , but, for general  $\beta \in \mathcal{O}$ , this need not be true.

We have the following representations

$$(10.13) \quad S_n(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \tilde{Z}_{n,k}(\beta),$$

$$(10.14) \quad S_n^\circ(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \tilde{Z}_{n,k}(\beta) - N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}}].$$

For  $T \in \mathbb{N}$ , define truncated versions of  $S_n(\beta)$  and  $S_n^\circ(\beta)$  by

$$(10.15) \quad S_{n,T}(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta),$$

$$(10.16) \quad S_{n,T}^\circ(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta) - N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T}].$$

The random function  $\tilde{Z}_{n,k}(\beta)$  is the  $(d-1)$ -level analogue of  $e^{-c_n(\beta)} \mathcal{Z}_n(\beta)$ , where  $c_n(\beta)$  is defined by (2.25). Note that  $\tilde{Z}_{n,k}(\beta)$  corresponds to a GREM with branching exponents  $(\alpha_2, \dots, \alpha_d)$  and variances  $(a_2, \dots, a_d)$ . Since we assumed that Proposition 10.2 (and hence, Corollary 10.4) is valid for any GREM with  $d-1$  levels, we have the following induction assumption. Fix some  $r \in (0, 2)$ .

(IND1) Let  $K'$  be a compact subset of  $E_2 \cap \{0 \leq \sigma < \frac{\sigma_2}{r}\}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K'$  and all  $n \in \mathbb{N}$ ,

$$\mathbb{E}|\tilde{Z}_{n,k}(\beta)|^r < C.$$

(IND2) Let  $K'$  be a compact subset of  $G^{d_1}E^{d-d_1} \cap \{0 \leq \sigma < \frac{\sigma_2}{r}\}$ , for some  $2 \leq d_1 \leq d$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K'$  and all  $n \in \mathbb{N}$ ,

$$\mathbb{E}|\tilde{Z}_{n,k}(\beta)|^r < C.$$

Note that (IND1) follows from Corollary 10.4 (which, as we have already shown, follows from Proposition 10.2), whereas (IND2) follows directly from Proposition 10.2, Part 2. Note that in the case  $d = 1$  (which is the basis of our induction), we have  $\tilde{Z}_{n,k}(\beta) = 1$  so that (IND1) is valid while (IND2) is empty. Equivalently, we can state (IND1) and (IND2) as follows:

(IND) Fix some  $r \in (0, 2)$ . Let  $K'$  be a compact subset of  $\mathcal{O} \cap \{0 \leq \sigma < \frac{\sigma_2}{r}\}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K'$  and all  $n \in \mathbb{N}$ ,

$$(10.17) \quad \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^r < C.$$

Our aim is to prove (10.5) and (10.6). This will be done in 4 steps.

STEP 1. In this step, we estimate the moments of  $S_{n,1}(\beta) = S_{n,1}^\circ(\beta)$ .

**Lemma 10.5.** *Fix  $p \in (0, 2)$ . Let  $K$  be a compact subset of  $\mathcal{O} \cap \{0 \leq \sigma < \frac{\sigma_1}{p}\}$ . Then, there is a constant  $C = C(K) > 0$  such that for all  $\beta \in K$  and all  $n \in \mathbb{N}$ ,*

$$(10.18) \quad \mathbb{E}|S_{n,1}(\beta)|^p = \mathbb{E}|S_{n,1}^\circ(\beta)|^p < C.$$

*Proof.* For future use, note the inequality, valid for  $\beta \in K$ ,

$$(10.19) \quad N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma p}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1}] \cdot \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^p < C.$$

Indeed, by Lemma 9.5 (recall that  $K \subset \{\sigma p < \sigma_1\}$ ) we can estimate the first factor on the left-hand side by  $C$ . Also, by the induction assumption (10.17) we have  $\mathbb{E}|\tilde{Z}_{n,k}(\beta)|^p \leq C$  (recall that  $K \subset \mathcal{O}$  and  $K \subset \{0 \leq \sigma < \frac{\sigma_1}{p}\} \subset \{0 \leq \sigma < \frac{\sigma_2}{p}\}$ ).

CASE 1:  $0 < p \leq 1$ . Then, by Proposition 3.2 and (10.19),

$$\mathbb{E}|S_{n,1}(\beta)|^p \leq N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma p}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^p < C.$$

CASE 2:  $1 \leq p < 2$ . Then, by Proposition 3.4 and (10.19),

$$\begin{aligned} \mathbb{E}|S_{n,1}(\beta)|^p &\leq C N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma p}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^p + C |\mathbb{E}S_{n,1}(\beta)|^p \\ &< C + C |\mathbb{E}S_{n,1}(\beta)|^p. \end{aligned}$$

We need to estimate  $\mathbb{E}S_{n,1}(\beta)$ . We have

$$|\mathbb{E}S_{n,1}(\beta)| = N_{n,1} |\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1})| \cdot |\mathbb{E}\tilde{Z}_{n,k}(\beta)|.$$

By Lemma 9.5 (recall that  $K \subset \{\sigma < \frac{\sigma_1}{p}\} \subset \{\sigma < \sigma_1\}$ , since  $p \geq 1$ ), we can estimate the first factor on the left-hand side by  $C$ . Also, by the induction assumption (10.17) we have  $\mathbb{E}|\tilde{Z}_{n,k}(\beta)|^p \leq C$  (recall that  $K \subset \mathcal{O}$  and  $K \subset \{0 \leq \sigma < \frac{\sigma_1}{p}\} \subset \{0 \leq$

$\sigma < \frac{\sigma_2}{p}\}$ ). By Lyapunov's inequality (3.1) (recall that  $p \geq 1$ ), this implies that  $|\mathbb{E}\tilde{Z}_{n,k}(\beta)| \leq C$ . It follows that  $|\mathbb{E}S_{n,1}(\beta)| < C$ .  $\square$

**STEP 2.** The aim of this step is to obtain estimates for the  $p$ -th moments of  $S_n(\beta) - S_{n,T}(\beta)$  and  $S_n^\circ(\beta) - S_{n,T}^\circ(\beta)$ . The inequalities which will prove will be needed in Section 11.3. The main result of this step is Lemma 10.8.

**Lemma 10.6.** *Let  $K$  be a compact subset of  $G^{d_1}E^{d-d_1} \cap \{0 \leq \sigma < \sigma_2\}$ , where  $2 \leq d_1 \leq d$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,*

$$(10.20) \quad |\mathbb{E}(S_n(\beta) - S_{n,T}(\beta))| < CT e^{-\varepsilon n}.$$

**Remark 10.7.** In the case  $d_1 = 1$ , we will prove a weaker estimate  $|\mathbb{E}(S_n(\beta) - S_{n,T}(\beta))| < CT$ .

*Proof of Lemma 10.6 and Remark 10.7.* Let  $K$  be a compact subset of  $G^{d_1}E^{d-d_1} \cap \{0 \leq \sigma < \sigma_2\}$ , where  $1 \leq d_1 \leq d$ . The subsequent estimates are valid uniformly over  $\beta \in K$ . We have  $K \subset \{\sigma > \frac{\sigma_1}{2}, \sigma + |\tau| > \sigma_1\}$ , that is the first level of the GREM is in the glassy phase. Thus, we can apply Lemma 9.7 to obtain

$$|\mathbb{E}(S_n(\beta) - S_{n,T}(\beta))| = N_{n,1} |\mathbb{E}P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \geq T}| |\mathbb{E}\tilde{Z}_{n,k}(\beta)| \leq CT |\mathbb{E}\tilde{Z}_{n,k}(\beta)|.$$

We have to estimate  $\mathbb{E}\tilde{Z}_{n,k}(\beta)$ . By definition of  $\tilde{Z}_{n,k}(\beta)$ , see (10.12), we have

$$(10.21) \quad \mathbb{E}\tilde{Z}_{n,k}(\beta) = \prod_{l=2}^d \left( e^{-c_{n,l}(\beta)} N_{n,l} e^{\frac{1}{2}\beta^2 a_l n} \right).$$

Note that the formula for  $c_{n,l}(\beta)$  depends on whether  $\beta \in G_l$  or  $\beta \in E_l$ ; see (10.2).

**CASE 1:**  $d_1 = 1$ . Then,  $\beta \in E_l$  for  $2 \leq l \leq d$ . With other words, the levels  $2, \dots, d$  are normalized by expectation; see (10.2). Hence, all terms in (10.21) are equal to 1 and  $\mathbb{E}\tilde{Z}_{n,k}(\beta) = 1$ . This proves Remark 10.7.

**CASE 2:**  $2 \leq d_1 \leq d$ . For  $d_1 < l \leq d$ , we have  $\beta \in E_l$  and hence, the corresponding factor in (10.21) is equal to 1; see (2.24). However, there is at least one  $l$  with  $2 \leq l \leq d_1$ . For such  $l$ , we have  $\beta \in G_l$  and (2.24) yields

$$|e^{-c_{n,l}(\beta)} N_{n,l} e^{\frac{1}{2}\beta^2 a_l n}| = e^{-\sigma \sqrt{n a_l} u_{n,l} + \log N_{n,l} + \frac{1}{2}(\sigma^2 - \tau^2) a_l n} = e^{\frac{1}{2} n a_l ((\sigma_l - \sigma)^2 - \tau^2) + o(n)}.$$

In the last step, we used (1.1) and (2.23). By the assumption of the lemma, we have  $\sigma < \sigma_2 \leq \sigma_l$ . Also, it follows from  $\beta \in G_l$  that  $0 < \sigma_l - \sigma < |\tau|$ . So, the expression  $(\sigma_l - \sigma)^2 - \tau^2$  admits a strictly negative upper bound  $-\varepsilon$  on  $K$ . Hence, for every  $2 \leq l \leq d_1$  (and there is at least one such  $l$ ) we can estimate the right-hand side by  $Ce^{-\varepsilon n}$ . It follows that, for  $2 \leq d_1 \leq d$ , we have the estimate

$$|\mathbb{E}\tilde{Z}_{n,k}(\beta)| < Ce^{-\varepsilon n}.$$

This completes the proof of (10.20).  $\square$

**Lemma 10.8.** *Fix  $p \in (0, 2)$ .*

- (1) *Let  $K$  be a compact subset of  $E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \frac{\sigma_2}{p}\}$ . There exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ :*

$$(10.22) \quad \mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^p \leq CT^{-\varepsilon}.$$

- (2) Let  $K$  be a compact subset of  $G^{d_1}E^{d-d_1} \cap \{0 \leq \sigma < \frac{\sigma_2}{p}\}$ , where  $2 \leq d_1 \leq d$ . There exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ :

$$(10.23) \quad \mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^p \leq CT^{-\varepsilon} + CT^2e^{-\varepsilon n}.$$

**Remark 10.9.** Under the assumptions of Part 2, there exists a constant  $C = C(K) > 0$  such that for all  $\beta \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$  we have

$$(10.24) \quad \mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^p \leq CT^p + CT^2e^{-\varepsilon n}.$$

To see this, note that by Lemma 9.7,

$$(10.25) \quad |(S_n^\circ(\beta) - S_{n,T}^\circ(\beta)) - (S_n(\beta) - S_{n,T}(\beta))| = N_{n,1}|\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T})| < CT.$$

Combining (10.23) and (10.25) and using Jensen's inequality (3.2), we obtain (10.24).

*Proof of Lemma 10.8.* We prove both parts of the lemma simultaneously. Write  $D = E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \frac{\sigma_2}{p}\}$  in the setting of Part 1 and  $D = G^{d_1}E^{d-d_1} \cap \{0 \leq \sigma < \frac{\sigma_2}{p}\}$  in the setting of Part 2. Consider some  $\beta_* = \sigma_* + i\tau_* \in K$ . So,  $\beta_* \in D$  and  $\frac{\sigma_1}{2} < \sigma_* < \frac{\sigma_2}{p}$ . Hence, there exists a closed disk  $U \subset D$  centered at  $\beta_*$  and a number  $q$  such that for all  $\beta = \sigma + i\tau \in U$ ,

$$(10.26) \quad \max\left\{p, \frac{\sigma_1}{\sigma}\right\} < q < \min\left\{\frac{\sigma_2}{\sigma}, 2\right\}.$$

In (10.22) and (10.23), it suffices to provide estimates for the  $q$ -th moment instead of the  $p$ -th moment since by Lyapunov's inequality (3.1) we have (recalling that  $p < q$ )

$$\begin{aligned} \mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^p &\leq (\mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^q)^{p/q}, \\ \mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^p &\leq (\mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^q)^{p/q} \end{aligned}$$

and since by Jensen's inequality (3.2),

$$(10.27) \quad (T^{-\varepsilon} + T^2e^{-\varepsilon n})^{p/q} \leq T^{-\varepsilon p/q} + T^{2p/q}e^{-\varepsilon pn/q} \leq T^{-\varepsilon p/q} + T^2e^{-\varepsilon pn/q}.$$

Also, note that it suffices to prove the required estimates (10.22) and (10.23) for  $\beta \in U$  since  $K$ , being a compact set, can be covered by finitely many  $U$ 's. In the sequel, we always take  $\beta \in U$ .

For future use, note that there exist  $C = C(K) > 0$ ,  $\varepsilon = \varepsilon(K) > 0$  such that for all  $\beta \in U$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,

$$(10.28) \quad N_{n,1}\mathbb{E}[P_{n,k}^{-\frac{\sigma q}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q < CT^{-\varepsilon}.$$

Indeed, by Lemma 9.6 (recall that  $U \subset \{\sigma q > \sigma_1\}$ ), we can estimate the first factor on the left-hand side by  $CT^{-\varepsilon}$ . Besides, by the induction assumption (10.17), we have  $\mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q \leq C$  (recall that  $U \subset \mathcal{O}$  and  $U \subset \{0 \leq \sigma < \frac{\sigma_2}{q}\}$ ).

**PART 1.** Assume that we are in the setting of Part 1 of Lemma 10.8. We prove (10.22).

**CASE 1:**  $0 < q \leq 1$ . Using Lemma 3.1, we obtain

$$(10.29) \quad \mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^q \leq C\mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^q + C|N_{n,1}\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T})|^q.$$

By Proposition 3.2 (which is applicable in the case  $0 < q \leq 1$ ) and by (10.28),

$$\mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^q \leq CN_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma q}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q \leq CT^{-\varepsilon}.$$

The second term on the right-hand side of (10.29) can also be estimated by  $CT^{-\varepsilon}$ . This is because we can apply Lemma 9.6 since  $U \subset \{\sigma q > \sigma_1\} \subset \{\sigma > \sigma_1\}$  by (10.26) and the assumption  $q \leq 1$ .

CASE 2:  $1 \leq q < 2$ . Recall that  $\mathbb{E}\tilde{Z}_{n,k}(\beta) = 1$  in the setting of Part 1. It follows that we can write

$$S_n^\circ(\beta) - S_{n,T}^\circ(\beta) = \sum_{k=1}^{N_{n,1}} (P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(\beta) - \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(\beta)]).$$

Note that the summands on the right-hand side have zero mean. By Proposition 3.3 (which is applicable in the case  $1 \leq q < 2$ ) and by Lemma 3.1 (where we use that  $q \geq 1$ ), we have

$$\begin{aligned} \mathbb{E}|S_n^\circ(\beta) - S_{n,T}^\circ(\beta)|^q &\leq CN_{n,1} \mathbb{E} \left| P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(\beta) - \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(\beta)] \right|^q \\ &\leq CN_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma q}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q. \end{aligned}$$

The right-hand side can be estimated by  $CT^{-\varepsilon}$  by (10.28).

PART 2. Assume that we are in the setting of Part 2 of Lemma 10.8. We prove (10.23). Note that  $\mathbb{E}\tilde{Z}_{n,k}(\beta)$  is not necessarily equal to 1 in the setting of Part 2. (In fact, only in phase  $G^1 E^{d-1}$  we have  $\mathbb{E}\tilde{Z}_{n,k}(\beta) = 1$ ).

CASE 1:  $0 < q \leq 1$ . By Proposition 3.2 and (10.28), we obtain that

$$(10.30) \quad \mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^q \leq N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma q}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q \leq CT^{-\varepsilon}.$$

CASE 2:  $1 \leq q < 2$ . By Proposition 3.4 (which is applicable in the case  $1 \leq q < 2$ ), we obtain that

$$\mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^q \leq CN_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\sigma q}{\sigma_1}} \mathbb{1}_{P_{n,k} > T}] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q + C|\mathbb{E}(S_n(\beta) - S_{n,T}(\beta))|^q.$$

The first summand on the right-hand side can be estimated by  $CT^{-\varepsilon}$  by (10.28). The second summand can be estimated by  $CT^q e^{-\varepsilon n} < CT^2 e^{-\varepsilon n}$  by Lemma 10.6. Let us show that the assumptions of this lemma are satisfied. We have  $U \subset G^{d_1} E^{d-d_1}$  with  $2 \leq d_1 \leq d$ . The assumption  $q \geq 1$ , together with (10.26), implies that  $U \subset \{0 \leq \sigma < \frac{\sigma_2}{q}\} \subset \{0 \leq \sigma < \sigma_2\}$ , so that we can indeed apply Lemma 10.6.

We have thus proved the estimates (10.22) and (10.23) for  $\beta \in U$ . By compactness we can cover  $K$  by a finite number of  $U$ 's. This completes the proof of Lemma 10.8.  $\square$

STEP 3. We are now ready to complete the proof of Proposition 10.2.

PART 1. Let  $K$  be a compact subset of  $E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \frac{\sigma_1}{p}\}$ . In Lemmas 10.5 and 10.8, Part 1, we proved the estimates  $\mathbb{E}|S_{n,1}^\circ(\beta)|^p < C$  and  $\mathbb{E}|S_n^\circ(\beta) - S_{n,1}^\circ(\beta)|^p < C$ . Using Jensen's inequality (3.2) we obtain that  $\mathbb{E}|S_n^\circ(\beta)|^p < C$ .

PART 2. Let  $K$  be a compact subset of  $G^{d_1} E^{d-d_1} \cap \{0 \leq \sigma < \frac{\sigma_1}{p}\}$ , where  $1 \leq d_1 \leq d$ . Note that

$$(10.31) \quad |S_n(\beta) - S_n^\circ(\beta)| = N_{n,1} |\mathbb{E}(P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}})| < C,$$

where the last step follows from Lemma 9.7.

CASE 1:  $d_1 = 1$ . Note that  $G^1 E^{d-1}$  is a subset of  $E_2$ . We already established in Part 1 of Proposition 10.2 that  $\mathbb{E}|S_n^\circ(\beta)|^p < C$ . From (10.31), we obtain that  $\mathbb{E}|S_n(\beta)|^p < C$ .

CASE 2:  $2 \leq d_1 \leq d$ . In Lemmas 10.5 and 10.8, Part 2, we proved the estimates  $\mathbb{E}|S_{n,1}(\beta)|^p < C$  and  $\mathbb{E}|S_n(\beta) - S_{n,1}(\beta)|^p < C$ . Using Jensen's inequality (3.2), we obtain that  $\mathbb{E}|S_n(\beta)|^p < C$ . It follows from (10.31) that  $\mathbb{E}|S_n^\circ(\beta)|^p < C$ , thus completing the proof of Proposition 10.2.

## 11. FUNCTIONAL LIMIT THEOREMS IN PHASES WITHOUT FLUCTUATION LEVELS

In this section, we prove functional limit theorems in phases of the form  $G^{d_1} E^{d-d_1}$ , where  $0 \leq d_1 \leq d$ . The proofs are based on the results of Section 10.

**11.1. Law of large numbers and absence of zeros in  $E_1$ .** We prove Theorems 2.19 and 2.23.

*Proof of Theorem 2.19.* We have to show that weakly on  $\mathcal{H}(E_1)$ ,

$$(11.1) \quad \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} \xrightarrow[n \rightarrow \infty]{w} 1.$$

For every fixed  $\beta \in E_1$ , the random variable  $\mathcal{Z}_n(\beta)/\mathbb{E}\mathcal{Z}_n(\beta)$  converges to 1 a.s. by Corollary 10.3 and the Borel–Cantelli lemma. Hence, (11.1) holds in the sense of finite-dimensional distributions. The tightness follows from Proposition 3.12 and Corollary 10.4.  $\square$

It is now easy to deduce Corollary 2.24. Indeed, applying Proposition 3.13 to (11.1), yields the desired weak convergence of  $\mathbf{Zeros}\{\mathcal{Z}_n(\beta): \beta \in E_1\}$  to the empty point process. To prove Theorem 2.23, we need a more refined argument.

*Proof of Theorem 2.23.* Let  $K$  be a compact subset of  $E_1$ . We have to prove that the probability that  $\mathcal{Z}_n(\beta)$  has at least one zero in  $K$  can be estimated by  $Ce^{-\varepsilon n}$ . Let  $\Gamma$  be a closed differentiable contour enclosing  $K$  and located in  $E_1$ . By the same argumentation as in Section 4.3 of [25], we have

$$\mathbb{P}[\exists \beta \in K: \mathcal{Z}_n(\beta) = 0] \leq C \oint_{\Gamma} \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right| |d\beta|.$$

Using Corollary 10.3 with  $p = 1$ , we obtain that there are  $C = C(\Gamma) > 0$  and  $\varepsilon = \varepsilon(\Gamma) > 0$  such that for every  $\beta \in \Gamma$  and every  $n \in \mathbb{N}$

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} - 1 \right| \leq C e^{-\varepsilon n}.$$

This yields the desired estimate.  $\square$

**11.2. Functional limit theorem in  $E_2 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$ .** The fluctuations of the random function  $\mathcal{Z}_n(\beta)$  in the domain  $\{|\sigma| < \frac{\sigma_1}{2}\}$  have been identified in Theorem 2.4 and in Section 7. In this section, we identify the fluctuations of  $\mathcal{Z}_n(\beta)$  in the domain  $E_2 \cap \{\sigma > \frac{\sigma_1}{2}\}$ .

**Theorem 11.1.** *Let  $D = E_2 \cap \{\sigma > \frac{\sigma_1}{2}\}$ . The following convergence of random analytic functions holds weakly on  $\mathcal{H}(D)$ :*

$$(11.2) \quad S_n^\circ(\beta) = \frac{\mathcal{Z}_n(\beta) - e^{\tilde{c}_n(\beta)} N_{n,1} \mathbb{E}[e^{\beta \sqrt{na_1} \xi} \mathbb{1}_{\xi < u_{n,1}}]}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P \left( \frac{\beta}{\sigma_1} \right) - \frac{\sigma_1}{\beta - \sigma_1}.$$

**Remark 11.2.** By symmetry, see (1.10), a similar result holds for the domain  $E_2 \cap \{\sigma < -\frac{\sigma_1}{2}\}$ . Namely, the following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_2 \cap \{\sigma < -\frac{\sigma_1}{2}\})$ :

$$(11.3) \quad \frac{\mathcal{Z}_n(\beta) - e^{\tilde{c}_n(-\beta)} N_{n,1} \mathbb{E}[e^{-\beta \sqrt{na_1} \xi} \mathbb{1}_{\xi < u_{n,1}}]}{e^{-\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(-\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P^- \left( -\frac{\beta}{\sigma_1} \right) + \frac{\sigma_1}{\beta + \sigma_1},$$

where  $\zeta_P^-$  is a copy of  $\zeta_P$ . In fact, one can even show that (11.2) and (11.3) can be combined into a *joint* convergence on the domain  $E_2 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$  and that the limiting functions  $\zeta_P$  and  $\zeta_P^-$  are *independent*. We will not provide a complete proof of the independence, but let us explain the idea. The function  $\zeta_P$  in (11.2) appears as the contribution of the *upper* extremal order statistics of the first GREM level. The function  $\zeta_P^-$  in (11.3) appears as the contribution of the *lower* extremal order statistics of the first GREM level. Since upper and lower extremal order statistics become independent in the large sample limit, we have the independence of  $\zeta_P$  and  $\zeta_P^-$ .

Let us stress that the domain  $E_2 \cap \{\sigma > \frac{\sigma_1}{2}\}$  on which Theorem 11.1 is valid includes the domain  $E_1 \cap \{\sigma > \frac{\sigma_1}{2}\}$ , the domain  $G^1 E^{d-1} \cap \{\sigma > 0\}$ , as well as the beak shaped boundary between these two domains. Restricting Theorem 11.1 to these smaller domains, we obtain two important corollaries. The first corollary is a restatement of Theorem 2.21.

**Corollary 11.3.** *The following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_1 \cap \{\sigma > \frac{\sigma_1}{2}\})$ :*

$$(11.4) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P \left( \frac{\beta}{\sigma_1} \right).$$

**Corollary 11.4.** *The following convergence of random analytic functions holds weakly on  $\mathcal{H}(G^1 E^{d-1} \cap \{\sigma > 0\})$ :*

$$(11.5) \quad \frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} = \frac{\mathcal{Z}_n(\beta)}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{w} \zeta_P \left( \frac{\beta}{\sigma_1} \right).$$



*Proof of Corollaries 11.3 and 11.4 given Theorem 11.1.* By Lemmas 9.3 and 9.4, locally uniformly on the specified domain, we have

$$\begin{aligned} e^{-\beta\sqrt{na_1}u_{n,1}} N_{n,1} \mathbb{E}[e^{\beta\sqrt{na_1}\xi} \mathbb{1}_{\xi < u_{n,1}}] &= \frac{\sigma_1}{\beta - \sigma_1}, & \text{if } \sigma + |\tau| > \sigma_1, \sigma > 0 \\ e^{-\beta\sqrt{na_1}u_{n,1}} N_{n,1} \mathbb{E}[e^{\beta\sqrt{na_1}\xi} \mathbb{1}_{\xi > u_{n,1}}] &= -\frac{\sigma_1}{\beta - \sigma_1}, & \text{if } \sigma + |\tau| < \sigma_1, \sigma > 0. \end{aligned}$$

Inserting this into (11.2), we immediately obtain (11.5) and (11.4).  $\square$

*Proof of Theorem 11.1.* First, we show that (11.2) holds in the sense of weak convergence of finite-dimensional distributions. Fix some  $\beta_1, \dots, \beta_r \in D$ . We continue to use the notation from Section 10. We are going to prove that the random vector  $\mathbf{S}_n^\circ := \{S_n^\circ(\beta_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_\infty^\circ = \{S_\infty^\circ(\beta_i)\}_{i=1}^r$ , where

$$\begin{aligned} S_n^\circ(\beta) &= \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \tilde{Z}_{n,k}(\beta) - N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}}], \\ S_\infty^\circ(\beta) &= \zeta_P\left(\frac{\beta}{\sigma_1}\right) - \frac{\sigma_1}{\beta - \sigma_1}. \end{aligned}$$

Note that this definition of  $S_n^\circ(\beta)$  is equivalent to the old ones; see (10.4), (10.14). We will verify the conditions of Lemma 3.15 for the random vectors  $\mathbf{S}_{n,T}^\circ := \{S_{n,T}^\circ(\beta_i)\}_{i=1}^r$  and  $\mathbf{S}_{\infty,T}^\circ := \{S_{\infty,T}^\circ(\beta_i)\}_{i=1}^r$ , where  $T \in \mathbb{N}$  is a truncation parameter and

$$\begin{aligned} S_{n,T}^\circ(\beta) &= \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta) - N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T}], \\ S_{\infty,T}^\circ(\beta) &= \sum_{k=1}^{\infty} P_k^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_k \leq T} - \int_1^T t^{-\frac{\beta}{\sigma_1}} dt. \end{aligned}$$

The three conditions of Lemma 3.15 will be verified in three steps.

STEP 1. We prove that  $\mathbf{S}_{n,T}^\circ \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}^\circ$  for every fixed  $T \in \mathbb{N}$ . By Lemma 9.1, we have the convergence of regularizing terms: for every  $\beta \in \mathbb{C}$ ,

$$(11.6) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T}] = \int_1^T t^{-\frac{\beta}{\sigma_1}} dt.$$

Since  $\tilde{Z}_{n,k}(\beta)$ , as defined in (10.12), is a  $(d-1)$ -level analogue of  $\mathcal{Z}_n(\beta)/\mathbb{E}\mathcal{Z}_n(\beta)$  for  $\beta \in E_2$ , we have that by Theorem 2.19, the random variable  $\tilde{Z}_{n,k}(\beta)$  converges in distribution to 1, for every  $\beta \in E_2$ . In particular, the random vector  $\tilde{\mathbf{Z}}_{n,k} := \{\tilde{Z}_{n,k}(\beta_i)\}_{i=1}^r$  converges in distribution to the random vector  $\tilde{\mathbf{Z}}_k := \{1\}_{i=1}^r$ . By Lemma 9.8, we obtain that

$$(11.7) \quad \left\{ \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_i}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta_i) \right\}_{i=1}^r \xrightarrow[n \rightarrow \infty]{d} \left\{ \sum_{k=1}^{\infty} P_k^{-\frac{\beta_i}{\sigma_1}} \mathbb{1}_{P_k \leq T} \right\}_{i=1}^r.$$

Combining (11.6) and (11.7), we obtain that  $\mathbf{S}_{n,T}^\circ \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}^\circ$ .

STEP 2. By [25, Theorem 2.6], see also (2.17), we have  $\mathbf{S}_{\infty,T}^\circ \xrightarrow[T \rightarrow \infty]{d} \mathbf{S}_\infty^\circ$ .

STEP 3. Let  $p \in (0, 2)$  be so close to 0 that  $\beta_1, \dots, \beta_r \in E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \frac{\sigma_1}{p}\}$ . To verify the second condition of Lemma 3.15 it suffices to prove that for every  $1 \leq i \leq r$  we have

$$(11.8) \quad \lim_{T \rightarrow \infty} \mathbb{E}|S_n^\circ(\beta_i) - S_{n,T}^\circ(\beta_i)|^p = 0 \text{ uniformly in } n \in \mathbb{N}.$$

However, this follows immediately from Lemma 10.8, Part 1.

STEP 4. Combining Steps 1, 2, 3 and applying Lemma 3.15, we obtain that the random vector  $\mathbf{S}_n^\circ$  converges in distribution to the random vector  $\mathbf{S}_\infty^\circ$ . Hence, (11.2) holds in the sense of weak convergence of finite-dimensional distributions. To complete the proof of Theorem 11.1 we need to show that the sequence of random functions  $S_n^\circ(\beta)$  is tight on  $\mathcal{H}(D)$ . By Proposition 3.14, it suffices to show that the sequence  $S_n^\circ(\beta)$  is tight on  $\mathcal{H}(U)$ , for arbitrary open set  $U$  such that  $\bar{U} \subset D$ . If  $p > 0$  is sufficiently small, then by Proposition 10.2, Part 1, there is a constant  $C$  such that  $\mathbb{E}|S_n^\circ(\beta)|^p < C$  for all  $\beta \in \bar{U}$  and all  $n \in \mathbb{N}$ . Proposition 3.12 implies that the sequence of random analytic functions  $S_n^\circ(\beta)$  is tight on  $\mathcal{H}(U)$ , thus completing the proof of Theorem 11.1.  $\square$

**11.3. Functional limit theorem in phase  $G^{d_1}E^{d-d_1}$ .** In this section, we prove Theorem 2.25. Fix some  $0 \leq d_1 \leq d$ . Denote by  $D$  the domain  $G^{d_1}E^{d-d_1} \cap \{\sigma > 0\}$ . Our aim is to show that weakly on  $\mathcal{H}(D)$ ,

$$(11.9) \quad S_n(\beta) = \frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \tilde{Z}_{n,k}(\beta) \xrightarrow[n \rightarrow \infty]{w} \zeta_P\left(\frac{\beta}{\sigma_1}, \dots, \frac{\beta}{\sigma_{d_1}}\right).$$

Note that in the case  $d_1 = 0$  (that is, in the phase  $E_1 = E^d$ ), the convergence in (11.9) (with the right-hand side interpreted as 1) has been established in Theorem 2.19. In the case  $d_1 = 1$ , we established (11.9) in Corollary 11.4.

We will use induction over  $d$ , the number of levels in the GREM. In the case  $d = 1$  (which is the basis of induction) we have  $d_1 = 0$  or  $d_1 = 1$ , so that (11.9) has already been established. We make the induction assumption that (11.9) holds for any GREM with  $d - 1$  levels. Our aim is to prove that it holds for the GREM with  $d$  levels. From now on, we may assume that  $d_1 \geq 2$ , that is at least *two* levels are in the glassy phase. We will use the notation

$$\beta^\Delta = \left(\frac{\beta}{\sigma_1}, \dots, \frac{\beta}{\sigma_{d_1}}\right) \in \mathbb{C}^{d_1}, \quad \tilde{\beta}^\Delta = \left(\frac{\beta}{\sigma_2}, \dots, \frac{\beta}{\sigma_{d_1}}\right) \in \mathbb{C}^{d_1-1},$$

First, we will show that (11.9) holds in the sense of weak convergence of finite-dimensional distributions. Fix some  $\beta_1, \dots, \beta_r \in D$ . Our aim is to prove that the random vector  $\mathbf{S}_n := \{S_n(\beta_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_\infty = \{S_\infty(\beta_i)\}_{i=1}^r$ , where

$$S_n(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \tilde{Z}_{n,k}(\beta), \quad S_\infty(\beta) = \zeta_P(\beta^\Delta).$$

This will be done by verifying the conditions of Lemma 3.15 for the random vectors  $\mathbf{S}_{n,T} := \{S_{n,T}(\beta_i)\}_{i=1}^r$  and  $\mathbf{S}_{\infty,T}(\beta) := \{S_{\infty,T}(\beta_i)\}_{i=1}^r$ , where  $T \in \mathbb{N}$  is a truncation

parameter and

$$S_{n,T}(\beta) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta), \quad S_{\infty,T}(\beta) = \sum_{k=1}^{\infty} P_k^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_k \leq T} \tilde{\zeta}_k(\tilde{\beta}^\Delta).$$

Here, we denote by  $\{\tilde{\zeta}_k(\tilde{\beta}^\Delta): k \in \mathbb{N}\}$  i.i.d. random analytic functions on  $D$  with the same law as  $\zeta_P(\tilde{\beta}^\Delta)$ .

STEP 1. We prove that  $\mathbf{S}_{n,T} \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}$  for every fixed  $T \in \mathbb{N}$ . The random function  $\tilde{Z}_{n,k}(\beta)$  is an analogue of the random function  $e^{-c_n(\beta)} \mathcal{Z}_n(\beta)$  with  $d-1$  levels. By the induction assumption, we have the following weak convergence on  $\mathcal{H}(D)$ :

$$\tilde{Z}_{n,k}(\beta) \xrightarrow[n \rightarrow \infty]{w} \zeta_P(\tilde{\beta}^\Delta),$$

From Lemma 9.8, it follows that

$$(11.10) \quad \left\{ \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_i}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(\beta_i) \right\}_{i=1}^r \xrightarrow[n \rightarrow \infty]{d} \left\{ \sum_{k=1}^{\infty} P_k^{-\frac{\beta_i}{\sigma_1}} \mathbb{1}_{P_k \leq T} \tilde{\zeta}_k(\tilde{\beta}_i^\Delta) \right\}_{i=1}^r.$$

This yields the desired convergence.

STEP 2. By Proposition 8.3, we have  $\mathbf{S}_{\infty,T} \xrightarrow[T \rightarrow \infty]{d} \mathbf{S}_\infty$  (recall that  $d_1 \geq 2$ ).

STEP 3. Fix  $\beta \in D$ . Let  $p > 0$  be so small that  $\sigma < \frac{\sigma_2}{p}$ . To verify the second condition of Lemma 3.15 it suffices to prove that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}|S_n(\beta) - S_{n,T}(\beta)|^p = 0.$$

However, this has already been established in Lemma 10.8, Part 2.

STEP 4. It follows from Steps 1, 2, 3 and Lemma 3.15, that the random vector  $\mathbf{S}_n$  converges in distribution to the random vector  $\mathbf{S}_\infty$ . In other words, (11.9) holds in the sense of weak convergence of finite-dimensional distributions. To complete the proof of Theorem 2.25 it remains to show that the sequence of random functions  $S_n(\beta)$  is tight on  $\mathcal{H}(D)$ . By Proposition 3.14, it suffices to show that the sequence  $S_n(\beta)$  is tight on  $\mathcal{H}(U)$ , for arbitrary open set  $U$  such that  $\bar{U} \subset D$ . If  $p > 0$  is sufficiently small, then by Proposition 10.2, Part 2, there is a constant  $C$  such that  $\mathbb{E}|S_n(\beta)|^p < C$  for all  $\beta \in \bar{U}$  and all  $n \in \mathbb{N}$ . Proposition 3.12 implies that the sequence of random analytic functions  $S_n(\beta)$  is tight on  $\mathcal{H}(U)$ , thus completing the proof of Theorem 2.25.

## 12. FUNCTIONAL LIMIT THEOREMS ON BEAK SHAPED BOUNDARIES

In this section, we prove Theorem 2.30, a functional limit theorem for the partition function  $\mathcal{Z}_n(\beta)$  in an infinitesimal neighborhood of some  $\beta_*$  located on the beak shaped boundary separating the phases  $G^{l-1}E^{d-l+1}$  and  $G^lE^{d-l}$ , for  $1 \leq l \leq d$ . The location and the size of the infinitesimal neighborhood are chosen to cover the “line of zeros” near the above mentioned boundary.

**12.1. Statement of the result and notation.** Fix some  $1 \leq l \leq d$ . Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that

$$(12.1) \quad \sigma_* > \frac{\sigma_l}{2}, \quad \tau_* > 0, \quad \sigma_* + \tau_* = \sigma_l.$$

These conditions imply that  $\beta_*$  belongs to the boundary separating the phases  $G^{l-1}E^{d-l+1}$  and  $G^lE^{d-l}$ . First we need to introduce several normalizing sequences. Let  $d_{n,l}$  be any complex sequence such that  $|d_{n,l}| = O(\log n)$  and

$$(12.2) \quad d_{n,l} + \beta_* \frac{\log(4\pi n \log \alpha_l)}{2\sigma_l} - ina_l \tau_*^2 \in 2\pi i\mathbb{Z}.$$

Let  $\beta_{n,l}(t)$  be a linear function of  $t$  which is given by

$$(12.3) \quad \beta_{n,l}(t) = \beta_* + e^{-\frac{3\pi i}{4}} \cdot \frac{1}{n} \cdot \frac{d_{n,l} + t}{\sqrt{2a_l \tau_*}}, \quad t \in \mathbb{C}.$$

Note that  $\lim_{n \rightarrow \infty} \beta_{n,l}(t) = \beta_*$  for all  $t \in \mathbb{C}$ . Note also that  $\operatorname{Re} d_{n,l} \sim -\frac{\sigma_*}{2\sigma_l} \log n$  is negative and hence,  $\beta_{n,l}(t)$  is located *outside*  $E_l$  provided that  $n$  is sufficiently large. The distance from  $\beta_{n,l}(t)$  to the boundary of  $E_l$  is asymptotic to  $\operatorname{const} \cdot \frac{\log n}{n}$ . Define a normalizing function

$$(12.4) \quad h_{n,l}(t) = \beta_{n,l}(t) \sum_{j=1}^l \sqrt{na_j} u_{n,j} + \sum_{j=l+1}^d \left( \log N_{n,j} + \frac{1}{2} \beta_{n,l}^2(t) na_j \right).$$

We can restate Theorem 2.30 as follows.

**Theorem 12.1.** *Fix some  $1 \leq l \leq d$  and some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  such that (12.1) holds. Then, weakly on  $\mathcal{H}(\mathbb{C})$  it holds that*

$$\left\{ \frac{\mathcal{Z}_n(\beta_{n,l}(t))}{e^{h_{n,l}(t)}} : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{e^t \zeta^{(l-1)} + \zeta^{(l)} : t \in \mathbb{C}\}.$$

Here,  $(\zeta^{(l-1)}, \zeta^{(l)})$  is a random vector given by

$$(\zeta^{(l-1)}, \zeta^{(l)}) = \left( \zeta_P \left( \frac{\beta_*}{\sigma_1}, \dots, \frac{\beta_*}{\sigma_{l-1}} \right), \zeta_P \left( \frac{\beta_*}{\sigma_1}, \dots, \frac{\beta_*}{\sigma_l} \right) \right),$$

where both zeta functions are based on the same Poisson cascade point process.

The remaining part of Section 12 is devoted to the proof of Theorem 12.1. We start by introducing the necessary notation. Define the random variables  $P_{n,k}$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the first level of the GREM) and  $\tilde{Z}_{n,k}(t)$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the remaining  $d-1$  levels of the GREM) by

$$(12.5) \quad P_{n,k} = e^{-\sigma_1 \sqrt{na_1}(\xi_k - u_{n,1})},$$

$$(12.6) \quad \tilde{Z}_{n,k}(t) = e^{-\tilde{h}_{n,l}(t)} \sum_{\tilde{\varepsilon} \in \tilde{\mathbb{S}}_n} e^{\beta_{n,l}(t) \sqrt{n}(\sqrt{a_2} \xi_{k\varepsilon_2} + \dots + \sqrt{a_d} \xi_{k\varepsilon_2 \dots \varepsilon_d})},$$

where  $\tilde{\mathbb{S}}_n$ , the index set for the levels  $2, \dots, d$ , is defined as in (10.10) and

$$(12.7) \quad \tilde{h}_{n,l}(t) = \beta_{n,l}(t) \sum_{j=2}^l \sqrt{na_j} u_{n,j} + \sum_{j=l+1}^d \left( \log N_{n,j} + \frac{1}{2} \beta_{n,l}^2(t) na_j \right).$$

By the definition of the GREM, these random variables have the following properties, for every  $n \in \mathbb{N}$ :

- (1)  $\tilde{Z}_{n,k}(t)$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random processes.
- (2)  $P_{n,k}$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random variables.
- (3) These two collections are independent.

The properties of  $P_{n,k}$  have been studied in Section 9. We have the representation

$$(12.8) \quad S_n(t) := \frac{\mathcal{Z}_n(\beta_{n,l}(t))}{e^{h_{n,l}(t)}} = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \tilde{Z}_{n,k}(t).$$

Introduce also a version of  $S_n(t)$  centered by a truncated expectation:

$$(12.9) \quad S_n^\circ(t) = S_n(t) - N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}} \right] \mathbb{E}[\tilde{Z}_{n,k}(t)].$$

For  $T \in \mathbb{N}$ , consider the truncated versions of  $S_n(t)$  and  $S_n^\circ(t)$  defined by

$$(12.10) \quad S_{n,T}(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(t),$$

$$(12.11) \quad S_{n,T}^\circ(t) = S_{n,T}(t) - N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T} \right] \mathbb{E}[\tilde{Z}_{n,k}(t)].$$

**12.2. Basis of induction:**  $l = 1$ . Our proof of Theorem 12.1 uses induction over  $l$ . First, we show that Theorem 12.1 holds for  $l = 1$ . Fix some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  such that  $\sigma_* > \frac{\sigma_1}{2}$ ,  $\tau_* > 0$ ,  $\sigma_* + \tau_* = \sigma_1$ . We are going to show that weakly on  $\mathcal{H}(\mathbb{C})$  it holds that

$$(12.12) \quad \left\{ \frac{\mathcal{Z}_n(\beta_{n,1}(t))}{e^{h_{n,1}(t)}} : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ e^t + \zeta_P \left( \frac{\beta_*}{\sigma_1} \right) : t \in \mathbb{C} \right\}.$$

The main step in the proof of (12.12) is the following result. Recall that  $\tilde{c}_n(\beta)$  was defined in (10.1) and (10.2).

**Proposition 12.2.** *The following convergence of random analytic functions holds weakly on  $\mathcal{H}(E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \sigma_1\})$ :*

$$\left\{ \frac{\mathcal{Z}_n(\beta)}{e^{\beta \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta)}} - N_{n,1} e^{\frac{1}{2} \beta^2 na_1 - \beta \sqrt{na_1} u_{n,1}} : \beta \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \left\{ \zeta_P \left( \frac{\beta}{\sigma_1} \right) : \beta \in \mathbb{C} \right\}.$$

*Proof.* We will use Theorem 11.1. It follows from the definition of  $P_{n,k}$ , see (12.5), that

$$\mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}}] = e^{\frac{1}{2} \beta^2 na_1 - \beta \sqrt{na_1} u_{n,1}}.$$

Using this and then Lemma 9.4, we obtain that locally uniformly on  $\{\sigma > 0, \sigma - |\tau| < \sigma_1\}$ ,

$$(12.13) \quad \begin{aligned} N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1}] &= N_{n,1} e^{\frac{1}{2} \beta^2 na_1 - \beta \sqrt{na_1} u_{n,1}} - N_{n,1} \mathbb{E}[P_{n,k}^{-\frac{\beta}{\sigma_1}} \mathbb{1}_{P_{n,k} < 1}] \\ &= N_{n,1} e^{\frac{1}{2} \beta^2 na_1 - \beta \sqrt{na_1} u_{n,1}} + \frac{\sigma_1}{\beta - \sigma_1} + o(1). \end{aligned}$$

In particular, this holds locally uniformly on  $E_2 \cap \{\frac{\sigma_1}{2} < \sigma < \sigma_1\}$ . Inserting (12.13) into Theorem 11.1 we obtain Proposition 12.2.  $\square$

**Lemma 12.3.** *If  $\beta_{n,l}(t)$  is given by (12.3), with some  $1 \leq l \leq d$ , then locally uniformly in  $t \in \mathbb{C}$ ,*

$$(12.14) \quad \lim_{n \rightarrow \infty} N_{n,l} e^{\frac{1}{2}\beta_{n,l}^2(t)na_l - \beta_{n,l}(t)\sqrt{na_l}u_{n,l}} = e^t.$$

*Proof.* The proof is a straightforward but lengthy calculation. Write  $\beta_n = \beta_* + \frac{\delta_n}{n}$ , where  $\delta_n = O(\log n)$  is some complex sequence. Using (1.1) and (2.23) we have

$$\begin{aligned} & \log N_{n,l} + \frac{na_l}{2}\beta_n^2 - \beta_n\sqrt{na_l}u_{n,l} \\ &= n \log \alpha_l + \frac{na_l}{2} \left( \beta_*^2 + 2\beta_* \frac{\delta_n}{n} \right) - \beta_*\sqrt{na_l} \left( \sqrt{2n \log \alpha_l} - \frac{\log(4\pi n \log \alpha_l)}{2\sqrt{2n \log \alpha_l}} \right) \\ & \quad - \frac{\delta_n}{n} \sqrt{na_l} \sqrt{2n \log \alpha_l} + o(1) \\ &= \delta_n a_l (\beta_* - \sigma_l) - na_l i \tau_*^2 + \beta_* \frac{\log(4\pi n \log \alpha_l)}{2\sigma_l} + o(1). \end{aligned}$$

In the last step, we used that  $\sigma_l \sqrt{a_l} = \sqrt{2 \log \alpha_l}$  and

$$n \log \alpha_l + \frac{na_l}{2}\beta_*^2 - \beta_*\sqrt{na_l}\sqrt{2n \log \alpha_l} = ina_l \tau_*(\sigma_* - \sigma_l) = -ina_l \tau_*^2.$$

Let us now choose

$$\delta_n := \frac{d_{n,l} + t}{a_l(\beta_* - \sigma_l)} = e^{-\frac{3\pi i}{4}} \frac{d_{n,l} + t}{\sqrt{2}a_l \tau_*},$$

where  $d_{n,l}$  satisfies (12.2). Then,  $\beta_n = \beta_{n,l}(t)$  and

$$\lim_{n \rightarrow \infty} \exp \left( \log N_{n,l} + \frac{na_l}{2}\beta_n^2 - \beta_n\sqrt{na_l}u_{n,l} \right) = e^t.$$

This completes the proof of (12.14).  $\square$

*Proof of (12.12).* Taking  $\beta = \beta_{n,1}(t)$  in Proposition 12.2 and applying Lemma 3.17, we obtain that the process

$$\left\{ \frac{\mathcal{Z}_n(\beta_{n,1}(t))}{e^{\beta_{n,1}(t)\sqrt{na_1}u_{n,1} + \tilde{c}_n(\beta_{n,1}(t))}} - N_{n,1} e^{\frac{1}{2}\beta_{n,1}^2(t)na_1 - \beta_{n,1}(t)\sqrt{na_1}u_{n,1}} : t \in \mathbb{C} \right\}$$

converges weakly on  $\mathcal{H}(\mathbb{C})$  to the process

$$\left\{ \zeta_P \left( \frac{\beta_*}{\sigma_1} \right) : t \in \mathbb{C} \right\}.$$

Note that the limit is considered as a stochastic process indexed by  $t \in \mathbb{C}$  (although it actually does not depend on  $t$ ). Applying Lemma 12.3, we obtain (12.12).  $\square$

The next lemma provides a moment estimate valid for  $\mathcal{Z}_n(\beta_{n,l}(t))$  in the case  $l = 1$ . It will serve as a basis of induction in the proof of Proposition 12.5.

**Lemma 12.4.** *Fix  $p \in (0, 2)$  such that  $p < \frac{\sigma_1}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,*

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta_{n,1}(t))}{e^{h_{n,1}(t)}} \right|^p \leq C.$$

*Proof.* Since  $\beta_{n,1}(t)$  converges to  $\beta_* \in E_2 \cap \{\sigma > \frac{\sigma_1}{2}\}$  uniformly in  $t \in K$ , we can use Proposition 10.2, Part 1, to obtain that

$$\mathbb{E} \left| \frac{\mathcal{Z}_n(\beta_{n,1}(t))}{e^{h_{n,1}(t)}} - N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,1}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1} \right] \right|^p \leq C.$$

We have already shown in (12.13) and Lemma 12.3 that uniformly in  $t \in K$ ,

$$N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,1}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > 1} \right] \leq C.$$

Using Jensen's inequality (3.2), we obtain the statement.  $\square$

**12.3. Moment estimates.** In this section, we prove estimates for the moments of  $S_n(t)$ . The main results are Proposition 12.5 and Lemma 12.9.

**Proposition 12.5.** *Fix  $p \in (0, 2)$  such that  $p < \frac{\sigma_1}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,*

$$(12.15) \quad \mathbb{E}|S_n(t)|^p < C.$$

The rest of the section is devoted to the proof of Proposition 12.5. We will use induction over  $l$ . Note that the case  $l = 1$  (which is the base of our induction) has been verified in Lemma 12.4. Let us take  $l \geq 2$  and assume that Proposition 12.5 holds for all smaller values of  $l$ . The random function  $\tilde{Z}_{n,k}(t)$  is the analogue of  $S_n(t)$ , with  $d$  and  $l$  reduced by 1. Thus, our induction assumption reads as follows.

(IND) Fix some  $r \in (0, 2)$  such that  $r < \frac{\sigma_2}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,

$$(12.16) \quad \mathbb{E}|\tilde{Z}_{n,k}(t)|^r < C.$$

STEP 1. In this step, we estimate the moments of  $S_{n,1}(t) = S_{n,1}^\circ(t)$ .

**Lemma 12.6.** *Fix  $p \in (0, 2)$  such that  $p < \frac{\sigma_1}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there is a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,*

$$(12.17) \quad \mathbb{E}|S_{n,1}^\circ(t)|^p = \mathbb{E}|S_{n,1}(t)|^p < C.$$

*Proof.* For future use, note the inequality, valid uniformly for  $t \in K$ ,

$$(12.18) \quad N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{p \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1} \right] \cdot \mathbb{E}|\tilde{Z}_{n,k}(t)|^p < C.$$

Here is a proof of (12.18). Note that  $p \operatorname{Re} \beta_{n,l}(t)$  converges to  $\sigma_* p < \sigma_1$  uniformly in  $t \in K$ . By Lemma 9.5, we can estimate the first factor on the left-hand side of (12.18) by  $C$ . Also, by the induction assumption (12.16) we have  $\mathbb{E}|\tilde{Z}_{n,k}(t)|^p \leq C$  (recall that we assume that  $p < \frac{\sigma_1}{\sigma_*} < \frac{\sigma_2}{\sigma_*}$ ). This proves (12.18).

CASE 1:  $0 < p \leq 1$ . Then, by Proposition 3.2 and (12.18),

$$\mathbb{E}|S_{n,1}(t)|^p \leq N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{p \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1} \right] \mathbb{E}|\tilde{Z}_{n,k}(t)|^p < C.$$

CASE 2:  $1 \leq p < 2$ . Then, by Proposition 3.4 and (12.18),

$$\begin{aligned} \mathbb{E}|S_{n,1}(t)|^p &\leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{p \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1} \right] \mathbb{E}|\tilde{Z}_{n,k}(t)|^p + C|\mathbb{E}S_{n,1}(t)|^p \\ &< C + C|\mathbb{E}S_{n,1}(t)|^p. \end{aligned}$$

We need to estimate  $\mathbb{E}S_{n,1}(t)$ . Clearly,

$$|\mathbb{E}S_{n,1}(t)| = N_{n,1} \left| \mathbb{E} \left( P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq 1} \right) \right| \cdot |\mathbb{E}\tilde{Z}_{n,k}(t)|.$$

Recall that  $\beta_{n,l}(t)$  converges to  $\beta_*$  uniformly in  $t \in K$  and note that  $\sigma_* < \frac{\sigma_1}{p} < \sigma_1$  because  $p \geq 1$ . By Lemma 9.5, we can estimate the first factor on the left-hand side by  $C$ . By the induction assumption (12.16) we have the estimate  $\mathbb{E}|\tilde{Z}_{n,k}(t)|^p \leq C$  (recall that  $p < \frac{\sigma_1}{\sigma_*} < \frac{\sigma_2}{\sigma_*}$ ). By Lyapunov's inequality (3.1) (recall that  $p \geq 1$ ), this implies that  $|\mathbb{E}\tilde{Z}_{n,k}(t)| \leq C$ . Hence, we obtain the estimate  $|\mathbb{E}S_{n,1}(t)| < C$ .  $\square$

STEP 2. In this step, we obtain estimates for the  $p$ -th moments of  $S_n(t) - S_{n,T}(t)$  and  $S_n^\circ(t) - S_{n,T}^\circ(t)$ . The main result of this step is Lemma 12.9.

**Lemma 12.7.** *Fix some  $3 \leq l \leq d$  and let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that (12.1) holds and, additionally,  $\sigma_* < \sigma_2$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,*

$$(12.19) \quad |\mathbb{E}(S_n(t) - S_{n,T}(t))| < CT e^{-\varepsilon n}.$$

**Remark 12.8.** In the case  $l = 2$ , we will prove a weaker estimate  $|\mathbb{E}(S_n(t) - S_{n,T}(t))| < CT$ .

*Proof of Lemma 12.7 and Remark 12.8.* Fix some  $2 \leq l \leq d$ . The subsequent estimates are valid uniformly over  $t \in K$ . Since  $\beta_{n,l}(t)$  converges to  $\beta_*$  and since  $\sigma_* + |\tau_*| = \sigma_l > \sigma_1$ , we can apply Lemma 9.7 to obtain

$$|\mathbb{E}(S_n(t) - S_{n,T}(t))| = N_{n,1} \left| \mathbb{E} P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \geq T} \right| |\mathbb{E}\tilde{Z}_{n,k}(t)| \leq CT |\mathbb{E}\tilde{Z}_{n,k}(t)|.$$

We have to estimate  $\mathbb{E}\tilde{Z}_{n,k}(t)$ . By definition of  $\tilde{Z}_{n,k}(t)$ , see (12.6) and (12.7), we have

$$(12.20) \quad \mathbb{E}\tilde{Z}_{n,k}(t) = \prod_{j=2}^l \left( e^{-\beta_{n,l}(t)\sqrt{na_j}u_{n,j}} N_{n,j} e^{\frac{1}{2}\beta_{n,l}^2(t)na_j} \right).$$

Note that the terms with  $j > l$  are missing in the product because they are equal to 1.

CASE 1:  $l = 2$ . In this case, the product in (12.20) has just one term which, by Lemma 12.3, converges to  $e^t$  uniformly in  $t \in K$ . We can estimate this term by  $C$ , thus proving Remark 12.8.

CASE 2:  $3 \leq l \leq d$ . By Lemma 12.3, the last factor in (12.20) converges to  $e^t$  uniformly in  $t \in K$ . Thus, we can estimate the last factor by  $C$ . However, there is



at least one factor with  $j \geq 2$  and  $j < l$ . For the latter one, we have (recall (1.1) and (2.23))

$$\begin{aligned} |e^{-\beta_{n,l}(t)\sqrt{na_j}u_{n,j}}N_{n,j}e^{\frac{1}{2}\beta_{n,l}^2(t)na_j}| &= e^{na_j(-\sigma_*\sigma_j+\frac{1}{2}\sigma_j^2+\frac{1}{2}(\sigma_*^2-\tau_*^2))+o(n)} \\ &= e^{\frac{1}{2}na_j((\sigma_j-\sigma_*)^2-\tau_*^2)+o(n)}. \end{aligned}$$

Since  $\sigma_j - \sigma_* < \sigma_l - \sigma_* = \tau_*$  and  $\sigma_j - \sigma_* \geq \sigma_2 - \sigma_* > 0$ , we can estimate the term by  $e^{-\varepsilon n}$ , for some sufficiently small  $\varepsilon > 0$  and all sufficiently large  $n$ . For the right-hand side of (12.20) we obtain the estimate

$$|\mathbb{E}\tilde{Z}_{n,k}(t)| < Ce^{-\varepsilon n}.$$

This completes the proof of (12.19).  $\square$

**Lemma 12.9.** *Fix some  $2 \leq l \leq d$  and some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  satisfying (12.1). Let  $p \in (0, 2)$  be such that  $p < \frac{\sigma_2}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ .*

- (1) *If  $l = 2$ , then there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that, for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , we have*

$$(12.21) \quad \mathbb{E}|S_n^\circ(t) - S_{n,T}^\circ(t)|^p \leq CT^{-\varepsilon}.$$

- (2) *If  $3 \leq l \leq d$ , then there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that, for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , we have*

$$(12.22) \quad \mathbb{E}|S_n(t) - S_{n,T}(t)|^p \leq CT^{-\varepsilon} + CT^2e^{-\varepsilon n}.$$

**Remark 12.10.** Under the assumptions of Part 1, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$  we have

$$(12.23) \quad \mathbb{E}|S_n(t) - S_{n,T}(t)|^p \leq CT^p.$$

To see this, note that  $\beta_{n,l}(t)$  converges to  $\beta_*$  and that  $\sigma_* + |\tau_*| = \sigma_2 > \sigma_1$  so that we can use Lemma 9.7 to obtain the estimate

$$(12.24) \quad |(S_n(t) - S_{n,T}(t)) - (S_n^\circ(t) - S_{n,T}^\circ(t))| = N_{n,1} \left| \mathbb{E} \left( P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right) \right| < CT.$$

Combining (12.21) and (12.24) and using Jensen's inequality (3.2), we obtain (12.23).

*Proof of Lemma 12.9.* We prove both parts of the lemma simultaneously. The subsequent estimates hold uniformly in  $t \in K$ . Since  $p < 2$ ,  $p < \frac{\sigma_2}{\sigma_*}$ ,  $\sigma_1 < \sigma_2$ ,  $\sigma_* > \frac{\sigma_l}{2} > \frac{\sigma_1}{2}$ , there exists a number  $q$  such that

$$(12.25) \quad \max \left\{ p, \frac{\sigma_1}{\sigma_*} \right\} < q < \min \left\{ \frac{\sigma_2}{\sigma_*}, 2 \right\}.$$

In (12.21) and (12.22), it suffices to provide estimates for the  $q$ -th moment instead of the  $p$ -th moment since by Lyapunov's inequality (3.1) we have (recalling that  $p < q$ )

$$\begin{aligned} \mathbb{E}|S_n(t) - S_{n,T}(t)|^p &\leq (\mathbb{E}|S_n(t) - S_{n,T}(t)|^q)^{p/q}, \\ \mathbb{E}|S_n^\circ(t) - S_{n,T}^\circ(t)|^p &\leq (\mathbb{E}|S_n^\circ(t) - S_{n,T}^\circ(t)|^q)^{p/q}. \end{aligned}$$

For future use, note that there exist  $C = C(K) > 0$ ,  $\varepsilon = \varepsilon(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n > n_0$ ,

$$(12.26) \quad N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(\beta)|^q < CT^{-\varepsilon}.$$

Indeed, by Lemma 9.6 (recall that  $\operatorname{Re} \beta_{n,l}(t)$  converges to  $\sigma_*$  and  $\sigma_* q > \sigma_1$  by (12.25)) we can estimate the first factor on the left-hand side by  $CT^{-\varepsilon}$ . By the induction assumption (12.16) we have  $\mathbb{E} |\tilde{Z}_{n,k}(\beta)|^q \leq C$  (recall that  $q < \frac{\sigma_2}{\sigma_*}$  by (12.25)). This proves (12.26).

PART 1. Assume that we are in the setting of Part 1 of Lemma 12.9. We prove (12.21). It follows from  $l = 2$  that we have the estimate  $|\mathbb{E} \tilde{Z}_{n,k}(t)| < C$ ; see Case 1 in the proof of Lemma 12.7.

CASE 1:  $0 < q \leq 1$ . Using Lemma 3.1, we obtain

$$(12.27) \quad \mathbb{E} |S_n^\circ(t) - S_{n,T}^\circ(t)|^q \leq C \mathbb{E} |S_n(t) - S_{n,T}(t)|^q + C \left| N_{n,1} \mathbb{E} \left( P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right) \right|^q |\mathbb{E} \tilde{Z}_{n,k}(t)|^q.$$

By Proposition 3.2 (which is applicable in the case  $0 < q \leq 1$ ) and by (12.26),

$$\mathbb{E} |S_n(t) - S_{n,T}(t)|^q \leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(\beta)|^q \leq CT^{-\varepsilon}.$$

The second term on the right-hand side of (12.27) can also be estimated by  $CT^{-\varepsilon}$ . Indeed, since  $\sigma_* > \frac{\sigma_1}{q} > \sigma_1$  by (12.25) and by the assumption  $0 < q \leq 1$ , we can apply Lemma 9.6 to obtain that

$$\left| N_{n,1} \mathbb{E} \left( P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right) \right| < CT^{-\varepsilon}.$$

Also, recall the estimate  $|\mathbb{E} \tilde{Z}_{n,k}(t)| < C$ .

CASE 2:  $1 \leq q < 2$ . It follows from (12.9) and (12.11) that we can write

$$S_n^\circ(t) - S_{n,T}^\circ(t) = \sum_{k=1}^{N_{n,1}} \left( P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(t) - \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(t) \right] \right).$$

The summands on the right-hand side have zero mean. By Proposition 3.3 (which is applicable in the case  $1 \leq q < 2$ ) and by Lemma 3.1 (where we use that  $q \geq 1$ ), we have

$$\begin{aligned} & \mathbb{E} |S_n^\circ(t) - S_{n,T}^\circ(t)|^q \\ & \leq CN_{n,1} \mathbb{E} \left| P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(t) - \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \tilde{Z}_{n,k}(t) \right] \right|^q \\ & \leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(t)|^q. \end{aligned}$$

The right-hand side can be estimated by  $CT^{-\varepsilon}$  by (12.26).

PART 2. Assume that we are in the setting of Part 2 of Lemma 12.9. We prove (12.22).

CASE 1:  $0 < q \leq 1$ . By Proposition 3.2 and (12.26), we obtain that

$$\mathbb{E}|S_n(t) - S_{n,T}(t)|^q \leq N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q \leq CT^{-\varepsilon}.$$

CASE 2:  $1 \leq q < 2$ . By Proposition 3.4 (which is applicable in the case  $1 \leq q < 2$ ), we obtain that

$$\begin{aligned} & \mathbb{E}|S_n(t) - S_{n,T}(t)|^q \\ & \leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_{n,l}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E}|\tilde{Z}_{n,k}(\beta)|^q + C|\mathbb{E}(S_n(t) - S_{n,T}(t))|^q \end{aligned}$$

The first summand on the right-hand side can be estimated by  $CT^{-\varepsilon}$  by (12.26), whereas the second summand can be estimated by  $CT^2 e^{-\varepsilon n}$  by Lemma 12.7. The assumptions of this lemma are satisfied because  $\sigma_* < \frac{\sigma_2}{q} < \sigma_2$  by (12.25) and the assumption  $1 \leq q < 2$ .

The proof of Lemma 12.9 is complete.  $\square$

**12.4. Proof of the functional limit theorem.** In this section, we prove Theorem 12.1. We have to show that weakly on  $\mathcal{H}(\mathbb{C})$ ,

$$(12.28) \quad S_n(t) = \frac{\mathcal{Z}_n(\beta_{n,l}(t))}{e^{h_{n,l}(t)}} = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \tilde{Z}_{n,k}(t) \xrightarrow[n \rightarrow \infty]{w} e^t \zeta^{(l-1)} + \zeta^{(l)}.$$

We will use induction over  $l$ . In the case  $l = 1$  (which is the basis of induction), we proved (12.28) in Section 12.2. Take some  $l \geq 2$  and assume that (12.28) has been established for all smaller values of  $l$ . The random function  $\tilde{Z}_{n,k}(t)$  is an analogue of the random function  $S_n(t)$  with  $d$  and  $l$  reduced by 1. By the induction assumption, we have the following weak convergence on  $\mathcal{H}(\mathbb{C})$ :

$$(12.29) \quad \{\tilde{Z}_{n,k}(t) : t \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \{e^t \tilde{\zeta}^{(l-2)} + \tilde{\zeta}^{(l-1)} : t \in \mathbb{C}\},$$

where

$$\tilde{\zeta}^{(l-2)} = \zeta_P \left( \frac{\beta_*}{\sigma_2}, \dots, \frac{\beta_*}{\sigma_{l-2}} \right), \quad \tilde{\zeta}^{(l-1)} = \zeta_P \left( \frac{\beta_*}{\sigma_2}, \dots, \frac{\beta_*}{\sigma_{l-1}} \right).$$

First, we will show that (12.28) holds in the sense of weak convergence of finite-dimensional distributions. Fix some  $t_1, \dots, t_r \in \mathbb{C}$ . We will prove that the random vector  $\mathbf{S}_n := \{S_n(t_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_\infty = \{S_\infty(t_i)\}_{i=1}^r$ , where

$$S_n(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,l}(t)}{\sigma_1}} \tilde{Z}_{n,k}(t), \quad S_\infty(t) = e^t \zeta^{(l-1)} + \zeta^{(l)}.$$

CASE A:  $3 \leq l \leq d$ . We will verify the conditions of Lemma 3.15 for the random vectors  $\mathbf{S}_{n,T} := \{S_{n,T}(t_i)\}_{i=1}^r$  and  $\mathbf{S}_{\infty,T} := \{S_{\infty,T}(t_i)\}_{i=1}^r$ , where  $T \in \mathbb{N}$  is a

truncation parameter and

$$S_{n,T}(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,1}(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(t),$$

$$S_{\infty,T}(t) = \sum_{k=1}^{\infty} P_k^{-\frac{\beta_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} (e^t \tilde{\zeta}_k^{(l-2)} + \tilde{\zeta}_k^{(l-1)}).$$

Here, we denote by  $(\tilde{\zeta}_k^{(l-2)}, \tilde{\zeta}_k^{(l-1)})$ ,  $k \in \mathbb{N}$ , independent copies of the random vector  $(\tilde{\zeta}^{(l-2)}, \tilde{\zeta}^{(l-1)})$ .

STEP A1. We prove that  $\mathbf{S}_{n,T} \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}$  for every  $T \in \mathbb{N}$ . From Lemma 9.8 and (12.29), it follows that

$$\left\{ \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_{n,1}(t_i)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(t_i) \right\}_{i=1}^r \xrightarrow[n \rightarrow \infty]{d} \left\{ \sum_{k=1}^{\infty} P_k^{-\frac{\beta_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} (e^{t_i} \tilde{\zeta}_k^{(l-2)} + \tilde{\zeta}_k^{(l-1)}) \right\}_{i=1}^r.$$

This is the desired convergence.

STEP A2. By Proposition 8.3, we have  $\mathbf{S}_{\infty,T} \xrightarrow[T \rightarrow \infty]{d} \mathbf{S}_{\infty}$  (at this point we use that  $l \geq 3$ ).

STEP A3. Fix  $t \in \mathbb{C}$ . To verify the third condition of Lemma 3.15, it suffices to prove that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} |S_n(t) - S_{n,T}(t)|^p = 0.$$

However, this has already been established in Lemma 12.9, Part 2. (Here, we again use that  $l \geq 3$ ).

CASE B:  $l = 2$ . We will prove that the random vector  $\mathbf{S}_n^{\circ} := \{S_n^{\circ}(t_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_{\infty}^{\circ} = \{S_{\infty}^{\circ}(t_i)\}_{i=1}^r$ , where

$$S_n^{\circ}(t) = S_n(t) - N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,2}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}} \right] \mathbb{E}[\tilde{Z}_{n,k}(t)],$$

$$S_{\infty}^{\circ}(t) = e^t \left( \zeta^{(1)} - \frac{\sigma_1}{\beta_* - \sigma_1} \right) + \zeta^{(2)}.$$

This implies that  $\mathbf{S}_n = \{S_n(t_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_{\infty} = \{S_{\infty}(t_i)\}_{i=1}^r$  because

$$(12.30) \quad \lim_{n \rightarrow \infty} N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,2}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k}} \right] = \frac{\sigma_1}{\beta_* - \sigma_1},$$

$$(12.31) \quad \lim_{n \rightarrow \infty} \mathbb{E}[\tilde{Z}_{n,k}(t)] = \lim_{n \rightarrow \infty} N_{n,2} e^{-\beta_{n,2}(t)\sqrt{na_2}u_{n,2} + \frac{1}{2}\beta_{n,2}^2(t)na_2} = e^t.$$

Note that (12.30) follows from Lemma 9.3, whereas (12.31) follows from (12.6), (12.7) and Lemma 12.3.

To prove that  $\mathbf{S}_n^{\circ}$  converges in distribution to  $\mathbf{S}_{\infty}^{\circ}$ , we will verify the conditions of Lemma 3.15 for the random vectors  $\mathbf{S}_{n,T}^{\circ} := \{S_{n,T}^{\circ}(t_i)\}_{i=1}^r$  and  $\mathbf{S}_{\infty,T}^{\circ} :=$

$\{S_{\infty,T}^\circ(t_i)\}_{i=1}^r$ , where  $T \in \mathbb{N}$  is a truncation parameter and

$$S_{n,T}^\circ(t) = S_{n,T}(t) - N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,2}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T} \right] \mathbb{E}[\tilde{Z}_{n,k}(t)],$$

$$S_{\infty,T}^\circ(t) = \sum_{k=1}^{\infty} P_k^{-\frac{\beta_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} (e^t + \zeta_k^{(1)}) - e^t \int_1^T y^{-\frac{\beta_*}{\sigma_1}} dy.$$

STEP B1. We prove that  $\mathbf{S}_{n,T}^\circ \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}^\circ$  for every  $T \in \mathbb{N}$ . In the same way as in Step A1 we have  $\mathbf{S}_{n,T} \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}$ . To complete the proof, recall (12.31) and note that by Lemma 9.1,

$$\lim_{n \rightarrow \infty} N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{\beta_{n,2}(t)}{\sigma_1}} \mathbb{1}_{1 \leq P_{n,k} \leq T} \right] = \int_1^T y^{-\frac{\beta_*}{\sigma_1}} dy.$$

This yields the desired convergence.

STEP B2. By Proposition 8.3 and (2.17), we have  $\mathbf{S}_{\infty,T}^\circ \xrightarrow[T \rightarrow \infty]{d} \mathbf{S}_{\infty}^\circ$ .

STEP B3. Fix  $t \in \mathbb{C}$ . To verify the third condition of Lemma 3.15, it suffices to prove that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} |S_n^\circ(t) - S_{n,T}^\circ(t)|^p = 0.$$

However, this has already been established in Lemma 12.9, Part 1.

Both in Case A and in Case B we showed that (12.28) holds in the sense of finite-dimensional distributions. To complete the proof of Theorem 12.1, we need to show that the sequence of random functions  $S_n(t)$  is tight on  $\mathcal{H}(\mathbb{C})$ . If  $p > 0$  is sufficiently small, then by Proposition 12.5 for every compact set  $K$  there exists a constant  $C = C(K) > 0$  such that  $\mathbb{E} |S_n(t)|^p < C$ , for all  $t \in K$  and all  $n \in \mathbb{N}$ . By Proposition 3.12, the sequence of random analytic functions  $S_n(t)$  is tight on  $\mathcal{H}(\mathbb{C})$ , thus completing the proof of Theorem 12.1.

**12.5. Functional limit theorem exactly on the boundary.** Fix some  $1 \leq l \leq d$  and take some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  such that (12.1) holds. In Theorem 12.1, we considered the fluctuations of  $\mathcal{Z}_n(\beta)$  in a small window located *outside*  $E_l$  at a distance of order  $\text{const} \cdot \frac{\log n}{n}$  from  $\beta_*$ . The distance was chosen so that the “line of zeros” becomes visible in the limit. In the sequel, we study what happens if we look at the partition function  $\mathcal{Z}_n(\beta_*)$  *exactly* on the beak shaped boundary of  $E_l$ . Define a sequence of normalizing constants

$$(12.32) \quad \hat{h}_{n,l}(\beta_*) = \sum_{j=1}^{l-1} \beta_* \sqrt{na_j} u_{n,j} + \sum_{j=l}^d \left( \log N_{n,j} + \frac{1}{2} \beta_*^2 na_j \right).$$

**Theorem 12.11.** *Fix some  $1 \leq l \leq d$  and some  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  such that (12.1) holds. Then,*

$$\frac{\mathcal{Z}_n(\beta_*)}{e^{\hat{h}_{n,l}(\beta_*)}} \xrightarrow[n \rightarrow \infty]{d} \zeta_P \left( \frac{\beta_*}{\sigma_1}, \dots, \frac{\beta_*}{\sigma_{l-1}} \right).$$

*Proof.* The proof of Theorem 12.11 uses the same method as the proof of Theorem 12.1, so we just describe the idea. We use induction over  $l$ . In the case  $l = 1$ , we already proved in Proposition 12.2 that

$$(12.33) \quad \frac{\mathcal{Z}_n(\beta_*) - \mathbb{E}\mathcal{Z}_n(\beta_*)}{e^{\beta_* \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta_*)}} \xrightarrow[n \rightarrow \infty]{d} \zeta_P \left( \frac{\beta_*}{\sigma_1} \right),$$

where we recall that  $\tilde{c}_n(\beta_*) = \sum_{j=2}^d (\log N_{n,j} + \frac{1}{2} \beta_*^2 n a_j)$ . By the same computation as in the proof of Lemma 12.3 with  $\delta_n = 0$ , we have

$$(12.34) \quad \frac{|\mathbb{E}\mathcal{Z}_n(\beta_*)|}{e^{\beta_* \sqrt{na_1} u_{n,1} + \tilde{c}_n(\beta_*)}} = |N_{n,1} e^{\frac{1}{2} \beta_*^2 n a_1 - \beta_* \sqrt{na_1} u_{n,1}}| = e^{\frac{\sigma_*}{2\sigma_1} \log(4\pi n \log \alpha_1) + o(1)},$$

which converges to  $+\infty$ . It follows from (12.33) and (12.34) that in the case  $l = 1$  we have

$$(12.35) \quad \frac{\mathcal{Z}_n(\beta_*)}{\mathbb{E}\mathcal{Z}_n(\beta_*)} \xrightarrow[n \rightarrow \infty]{d} 1.$$

This proves Theorem 12.11 in the case  $l = 1$ , thus establishing the basis of induction. The rest of the proof, namely the adjoining of glassy phase levels to (12.35), is analogous to the proof of Theorem 12.1.  $\square$

### 13. FUNCTIONAL LIMIT THEOREMS IN PHASES WITH AT LEAST ONE FLUCTUATION LEVEL

In this section, we prove functional limit theorems describing the local behavior of the partition function  $\mathcal{Z}_n(\beta)$  near some  $\beta_* = \sigma_* + i\tau_*$  located inside or on the boundary of the phase  $G^{d_1} F^{d_2} E^{d_3}$ , where  $d_2 \geq 1$ . Note that the case  $d_2 = 0$  has been considered in Section 11. Our main aim in this section is to prove Theorem 2.28. The proofs of Theorems 2.32, 2.35, 2.37, 2.38 are all very similar and will be discussed in Section 13.4.

**13.1. Notation.** Fix some  $d_1, d_2, d_3 \in \{0, \dots, d\}$  such that  $d_1 + d_2 + d_3 = d$  and  $d_2 \geq 1$ . Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that

$$(13.1) \quad \beta_* \in G^{d_1} F^{d_2} E^{d_3}, \quad \sigma_* \geq 0, \quad \tau_* > 0.$$

Define a local coordinate near  $\beta_*$  by

$$(13.2) \quad \beta_n(t) = \beta_* + \frac{t}{\sqrt{n}}, \quad t \in \mathbb{C}.$$

For  $1 \leq k \leq d$ , define the normalizing functions  $c_{n,k}(\beta_*; t)$ , where  $t \in \mathbb{C}$ , by

$$(13.3) \quad c_{n,k}(\beta_*; t) = \begin{cases} \beta_n(t) \sqrt{na_k} u_{n,k}, & \text{if } \beta \in G_k, \\ \frac{1}{2} \log N_{n,k} + a_k (\sqrt{n} \sigma_* + t)^2, & \text{if } \beta \in F_k, \\ \log N_{n,k} + \frac{1}{2} a_k (\sqrt{n} \beta_* + t)^2, & \text{if } \beta \in E_k. \end{cases}$$

Define also the normalizing functions

$$(13.4) \quad c_n(\beta_*; t) = c_{n,1}(\beta_*; t) + \dots + c_{n,d}(\beta_*; t),$$

$$(13.5) \quad \tilde{c}_n(\beta_*; t) = c_{n,2}(\beta_*; t) + \dots + c_{n,d}(\beta_*; t).$$

Note that these functions are linear or quadratic in  $t$ . Consider a random analytic function  $\{S_n(t) : t \in \mathbb{C}\}$  defined by

$$S_n(t) = \frac{\mathcal{Z}_n(\beta_n(t))}{e^{c_n(\beta_*; t)}}.$$

Our aim is to show that  $S_n(t)$  converges weakly on  $\mathcal{H}(\mathbb{C})$  and to identify the limiting process.

Define the random variables  $P_{n,k}$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the first level of the GREM) and  $\tilde{Z}_{n,k}(t)$ ,  $n \in \mathbb{N}$ ,  $1 \leq k \leq N_{n,1}$ , (the normalized contributions of the remaining  $d-1$  levels of the GREM) by

$$(13.6) \quad P_{n,k} = e^{-\sigma_1 \sqrt{na_1}(\xi_k - u_{n,1})},$$

$$(13.7) \quad \tilde{Z}_{n,k}(t) = e^{-\tilde{c}_n(\beta_*; t)} \sum_{\tilde{\varepsilon} \in \tilde{\mathbb{S}}_n} e^{\beta_n(t) \sqrt{n}(\sqrt{a_2} \xi_{k\varepsilon_2} + \dots + \sqrt{a_d} \xi_{k\varepsilon_2 \dots \varepsilon_d})},$$

where  $\tilde{\mathbb{S}}_n$  is as in (10.10). By the definition of the GREM, these random variables have the following properties, for every  $n \in \mathbb{N}$ :

- (1)  $\tilde{Z}_{n,k}(t)$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random processes.
- (2)  $P_{n,k}$ ,  $1 \leq k \leq N_{n,1}$ , is an i.i.d. collection of random variables.
- (3) These two collections are independent.

In the case  $d_1 \geq 1$ , we have the representation

$$(13.8) \quad S_n(t) = \frac{\mathcal{Z}_n(\beta_n(t))}{e^{c_n(\beta_*; t)}} = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_n(t)}{\sigma_1}} \tilde{Z}_{n,k}(t).$$

For  $T \in \mathbb{N}$ , define the truncated version of  $S_n(t)$  by

$$(13.9) \quad S_{n,T}(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(t).$$

**13.2. Moment estimates.** In this section, we prove estimates for the  $p$ -th moments of  $S_n(t)$  and  $S_{n,T}(t)$ . The main results are Proposition 13.1 and Lemma 13.4.

**Proposition 13.1.** *Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that (13.1) holds with some  $d_2 \geq 1$ . Fix  $p \in (0, 2)$  such that  $p < \frac{\sigma_1}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,*

$$(13.10) \quad \mathbb{E}|S_n(t)|^p < C.$$

*Proof.* We will use induction over  $d_1$ , the number of glassy phase levels. We have already verified the case  $d_1 = 0$  (which is the base of our induction) in (4.13). Indeed, if  $d_1 = 0$ , then for every compact set  $K \subset \mathbb{C}$  we can find  $c = c(K) > 0$  such that

$$\sqrt{\mathbb{E}|\mathcal{Z}_n(\beta_n(t))|^2} > \sqrt{\text{Var } \mathcal{Z}_n(\beta_n(t))} > ce^{c_n(\beta_*; t)}$$

by Proposition 6.1 (which holds uniformly in  $t \in K$ ) and hence,

$$\mathbb{E}|S_n(t)|^p = \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta_n(t))}{e^{c_n(\beta_*; t)}} \right|^p \leq C \mathbb{E} \left| \frac{\mathcal{Z}_n(\beta_n(t))}{\sqrt{\mathbb{E}|\mathcal{Z}_n(\beta_n(t))|^2}} \right|^p \leq C,$$

where the last step is by (4.13). Note at this point that although (4.13) is stated for  $2 < p < \frac{\sigma_1^2}{2\sigma_*^2}$  (this interval is non-empty for  $d_1 = 0$ ,  $d_2 \geq 1$ ), the same inequality continues to hold for  $0 < p \leq 2$  by the Lyapunov's inequality (3.1).

Let us therefore take  $d_1 \geq 1$  and assume that Proposition 13.1 holds in the setting of  $d_1 - 1$  glassy phase levels. The random function  $\tilde{Z}_{n,k}(t)$  is the analogue of  $S_n(t)$  with  $d_1 - 1$  glassy phase levels. Hence, our induction assumption reads as follows.

(IND) Fix some  $r \in (0, 2)$  such that  $r < \frac{\sigma_2}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exists a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,

$$(13.11) \quad \mathbb{E}|\tilde{Z}_{n,k}(t)|^r < C.$$

STEP 1. In this step, we estimate the moments of  $S_{n,1}(t)$ .

**Lemma 13.2.** *Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that (13.1) holds with some  $d_1 \geq 1$  and  $d_2 \geq 1$ . Fix  $p \in (0, 2)$  such that  $p < \frac{\sigma_1}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there is a constant  $C = C(K) > 0$  such that for all  $t \in K$  and all  $n \in \mathbb{N}$ ,*

$$(13.12) \quad \mathbb{E}|S_{n,1}(t)|^p < C.$$

*Proof.* The proof of Lemma 12.6 applies with straightforward changes.  $\square$

STEP 2. In this step, we obtain estimates for the  $p$ -th moment of  $S_n(t) - S_{n,T}(t)$ . The main result of this step is Lemma 13.4.

**Lemma 13.3.** *Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that (13.1) holds with some  $d_1 \geq 1$  and  $d_2 \geq 1$ . Assume, additionally, that  $\sigma_* < \sigma_2$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,*

$$(13.13) \quad |\mathbb{E}(S_n(t) - S_{n,T}(t))| < CT e^{-\varepsilon n}.$$

*Proof.* The subsequent estimates are valid uniformly over  $t \in K$ . Since  $\beta_n(t)$  converges to  $\beta_*$  and since  $\sigma_* + |\tau_*| > \sigma_1$  (because  $d_1 \geq 1$ ), we can apply Lemma 9.7 to obtain

$$|\mathbb{E}(S_n(t) - S_{n,T}(t))| = N_{n,1} \left| \mathbb{E} P_{n,k}^{-\frac{\beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \geq T} \right| |\mathbb{E} \tilde{Z}_{n,k}(t)| \leq CT |\mathbb{E} \tilde{Z}_{n,k}(t)|.$$

We need to estimate  $\mathbb{E} \tilde{Z}_{n,k}(t)$ . By definition of  $\tilde{Z}_{n,k}(t)$ , see (13.7) and (13.3), we have

$$(13.14) \quad \mathbb{E} \tilde{Z}_{n,k}(t) = \prod_{j=2}^{d_1} \left( N_{n,j} e^{\frac{1}{2} \beta_*^2 a_j n - \beta_* \sqrt{n a_j} u_{n,j}} \right) \prod_{j=d_1+1}^{d_1+d_2} \left( N_{n,j}^{1/2} e^{\frac{1}{2} \beta_*^2 a_j n - a_j \sigma_*^2 n} \right) \cdot e^{O(\sqrt{n})}.$$

Note that the factors with  $j > d_1 + d_2$  are missing on the right-hand side because they are equal to 1. We will show that every term in any product on the right-hand side can be estimated by  $e^{-\varepsilon n}$ , for sufficiently large  $n$ . Since  $d_2 \geq 1$ , there is a least one such term and we obtain the required estimate. Consider some term with  $2 \leq j \leq d_2$ :

$$N_{n,j} e^{\frac{1}{2} \beta_*^2 a_j n - \beta_* \sqrt{n a_j} u_{n,j}} = e^{\frac{1}{2} n a_j ((\sigma_j - \sigma_*)^2 - \tau_*^2) + o(n)} < e^{-\varepsilon n},$$

where the last estimate holds since  $\sigma_* < \sigma_2 \leq \sigma_j$  and  $\sigma_* + |\tau_*| > \sigma_j$  (because  $\beta \in G_j$ ). Consider some term with  $d_1 < j \leq d_1 + d_2$ :

$$|N_{n,j}^{1/2} e^{\frac{1}{2} \beta_*^2 a_j n - a_j \sigma_*^2 n}| = e^{\frac{1}{2} n a_j (\frac{1}{2} \sigma_j^2 - |\beta_*|^2)} < e^{-\varepsilon n},$$

where the last estimate holds since  $2|\beta_*|^2 > \sigma_j$  (because  $\beta_* \in F_j$ ).  $\square$



**Lemma 13.4.** *Let  $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$  be such that (13.1) holds with  $d_1 \geq 1$ ,  $d_2 \geq 1$ . Let  $p \in (0, 2)$  be such that  $p < \frac{\sigma_2}{\sigma_*}$ . Let  $K$  be a compact subset of  $\mathbb{C}$ . Then, there exist constants  $C = C(K) > 0$  and  $\varepsilon = \varepsilon(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ ,  $n \in \mathbb{N}$  we have*

$$(13.15) \quad \mathbb{E}|S_n(t) - S_{n,T}(t)|^p \leq CT^{-\varepsilon} + CT^2e^{-\varepsilon n}.$$

*Proof.* The subsequent estimates hold uniformly in  $t \in K$ . By the inequalities  $p < 2$ ,  $p < \frac{\sigma_2}{\sigma_*}$ ,  $\sigma_1 < \sigma_2$ ,  $\sigma_* > \frac{\sigma_1}{2}$ , there exists a number  $q$  such that

$$(13.16) \quad \max \left\{ p, \frac{\sigma_1}{\sigma_*} \right\} < q < \min \left\{ \frac{\sigma_2}{\sigma_*}, 2 \right\}.$$

By Lyapunov's inequality (3.1) (recall that  $p < q$ ), it suffices to establish (13.15) with  $p$ -th moment replaced by the  $q$ -th moment.

For future use, note that there exist  $C = C(K) > 0$ ,  $\varepsilon = \varepsilon(K) > 0$  such that for all  $t \in K$ ,  $T \in \mathbb{N}$ , and all sufficiently large  $n \in \mathbb{N}$ ,

$$(13.17) \quad N_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(t)|^q < CT^{-\varepsilon}.$$

We can prove (13.17) as follows. By Lemma 9.6 (recall that  $\operatorname{Re} \beta_n(t)$  converges to  $\sigma_*$  and  $\sigma_* q > \sigma_1$  by (13.16)), we can estimate the first factor on the left-hand side by  $CT^{-\varepsilon}$ . By the induction assumption (13.11), we have  $\mathbb{E} |\tilde{Z}_{n,k}(t)|^q \leq C$  (recall that  $q < \frac{\sigma_2}{\sigma_*}$  by (13.16)). We are now ready to prove (13.15).

CASE 1:  $0 < q \leq 1$ . By Proposition 3.2 (which is applicable in the case  $0 < q \leq 1$ ) and by (13.17),

$$\mathbb{E}|S_n(t) - S_{n,T}(t)|^q \leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(t)|^q \leq CT^{-\varepsilon}.$$

CASE 2:  $1 \leq q < 2$ . By Proposition 3.4 (which is applicable in the case  $1 \leq q < 2$ ), we obtain that

$$\begin{aligned} & \mathbb{E}|S_n(t) - S_{n,T}(t)|^q \\ & \leq CN_{n,1} \mathbb{E} \left[ P_{n,k}^{-\frac{q \operatorname{Re} \beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} > T} \right] \mathbb{E} |\tilde{Z}_{n,k}(t)|^q + C|\mathbb{E}(S_n(t) - S_{n,T}(t))|^q \end{aligned}$$

The first summand on the right-hand side can be estimated by  $CT^{-\varepsilon}$  by (13.17), whereas the second summand can be estimated by  $CT^2e^{-\varepsilon n}$  by Lemma 13.3. The assumptions of this lemma are satisfied because  $\sigma_* < \frac{\sigma_2}{q} < \sigma_2$  by (13.16) and the assumption  $1 \leq q < 2$ .

The proof of Lemma 13.4 is complete.  $\square$

To complete the proof of Proposition 13.1, combine the results of Lemmas 13.2 and 13.4.  $\square$

**13.3. Proof of the functional limit theorem.** In this section, we prove Theorem 2.28. Introduce the notation

$$\sigma_*^\Delta = \left( \frac{\sigma_*}{\sigma_1}, \dots, \frac{\sigma_*}{\sigma_{d_1}} \right) \in \mathbb{R}^{d_1}, \quad \tilde{\sigma}_*^\Delta = \left( \frac{\sigma_*}{\sigma_2}, \dots, \frac{\sigma_*}{\sigma_{d_1}} \right) \in \mathbb{R}^{d_1-1}.$$

Consider random analytic functions  $\{S_n(t) : t \in \mathbb{C}\}$  (which is the same as in (13.8)) and  $\{S_\infty(t) : t \in \mathbb{C}\}$  given by

$$S_n(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_n(t)}{\sigma_1}} \tilde{Z}_{n,k}(t) = \frac{\mathcal{Z}_n(\beta_n(t))}{e^{c_n(\beta_*; t)}},$$

$$S_\infty(t) = \sqrt{\zeta_P(2\sigma_*^\Delta)} \mathbb{X}(\kappa t),$$

where  $\{\mathbb{X}(t) : t \in \mathbb{C}\}$  is the plane Gaussian analytic function independent of  $\zeta_P$  and  $\kappa$  is the total variance of the fluctuation levels, as in Theorem 2.28. We can now state Theorem 2.28 as follows: Weakly on  $\mathcal{H}(\mathbb{C})$ ,

$$(13.18) \quad \{S_n(t) : t \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \{S_\infty(t) : t \in \mathbb{C}\}.$$

To prove (13.18), we will use induction over  $d_1$ , the number of glassy phase levels. In the case  $d_1 = 0$  (which is the basis of induction) we already established (13.18) in Theorem 7.1. Note that in the case  $d_1 = 0$  we have  $\sigma_*^\Delta = \emptyset$  and hence,  $\zeta_P(2\sigma_*^\Delta) = 1$  by convention. Take some  $d_1 \geq 1$  and assume that (13.18) has been established in the setting of  $d_1 - 1$  glassy phase levels. The random function  $\tilde{Z}_{n,k}(t)$  is an analogue of the random function  $S_n(t)$  with  $d_1$  reduced by 1. By the induction assumption, we have the following weak convergence on  $\mathcal{H}(\mathbb{C})$ :

$$(13.19) \quad \{\tilde{Z}_{n,k}(t) : t \in \mathbb{C}\} \xrightarrow[n \rightarrow \infty]{w} \left\{ \sqrt{\zeta_P(2\tilde{\sigma}_*^\Delta)} \mathbb{X}(\kappa t) : t \in \mathbb{C} \right\}.$$

First, we will show that (13.18) holds in the sense of weak convergence of finite-dimensional distributions. Fix some  $t_1, \dots, t_r \in \mathbb{C}$ . We will prove that the random vector  $\mathbf{S}_n := \{S_n(t_i)\}_{i=1}^r$  converges in distribution to  $\mathbf{S}_\infty = \{S_\infty(t_i)\}_{i=1}^r$ . We will verify the conditions of Lemma 3.15 for the random vectors  $\mathbf{S}_{n,T} := \{S_{n,T}(t_i)\}_{i=1}^r$  and  $\mathbf{S}_{\infty,T}(\beta) := \{S_{\infty,T}(t_i)\}_{i=1}^r$ , where  $T \in \mathbb{N}$  is a truncation parameter and

$$S_{n,T}(t) = \sum_{k=1}^{N_{n,1}} P_{n,k}^{-\frac{\beta_n(t)}{\sigma_1}} \mathbb{1}_{P_{n,k} \leq T} \tilde{Z}_{n,k}(t),$$

$$S_{\infty,T}(t) = \sum_{k=1}^{\infty} P_k^{-\frac{\beta_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} \sqrt{V_k} \mathbb{X}_k(\kappa t).$$

Here, we denote by  $V_k$ ,  $k \in \mathbb{N}$ , and  $\{\mathbb{X}_k(\kappa t) : t \in \mathbb{C}\}$ ,  $k \in \mathbb{N}$ , independent copies of the random variable  $\zeta_P(2\tilde{\sigma}_*^\Delta)$  and the random analytic function  $\{\mathbb{X}(\kappa t) : t \in \mathbb{C}\}$ .

STEP 1. We prove that  $\mathbf{S}_{n,T} \xrightarrow[n \rightarrow \infty]{d} \mathbf{S}_{\infty,T}$ . This follows from Lemma 9.8 and (13.19). Recall, in particular, that  $\beta_n(t)$  converges to  $\beta_*$ .

STEP 2. We prove that  $\mathbf{S}_{\infty,T} \xrightarrow[T \rightarrow \infty]{d} \mathbf{S}_\infty$ . Let  $\mathcal{A}_{P,V}$  be the  $\sigma$ -algebra generated by  $\{P_k : k \in \mathbb{N}\}$  and  $\{V_k : k \in \mathbb{N}\}$ . Conditioning on  $\mathcal{A}_{P,V}$  and treating  $P_k, V_k$ ,  $k \in \mathbb{N}$ , as constants we have

$$\begin{aligned} \{S_{\infty,T}(t) : t \in \mathbb{C}\} | \mathcal{A}_{P,V} &\stackrel{d}{=} \left\{ \sum_{k=1}^{\infty} P_k^{-\frac{\beta_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} \sqrt{V_k} \mathbb{X}_k(\kappa t) : t \in \mathbb{C} \right\} \Big| \mathcal{A}_{P,V} \\ &\stackrel{d}{=} \left\{ \sqrt{\zeta_P(2\sigma_*^\Delta; T)} \mathbb{X}(\kappa t) : t \in \mathbb{C} \right\} \Big| \mathcal{A}_{P,V} \end{aligned}$$

where  $\zeta_P(2\sigma_*^\Delta; T) = \sum_{k=1}^{\infty} P_k^{-\frac{2\sigma_*}{\sigma_1}} \mathbb{1}_{P_k \leq T} V_k$ . Integrating over  $P_k, V_k$ ,  $k \in \mathbb{N}$ , we obtain that

$$\{S_{\infty, T}(t) : t \in \mathbb{C}\} \stackrel{d}{=} \left\{ \sqrt{\zeta_P(2\sigma_*^\Delta; T)} \mathbb{X}(\kappa t) : t \in \mathbb{C} \right\}.$$

By Theorem 2.12, the random variable  $\zeta_P(2\sigma_*^\Delta; T)$  converges a.s. to  $\zeta_P(2\sigma_*)$ , as  $T \rightarrow \infty$ . This yields the statement of Step 2 and verifies the third condition of Lemma 3.15.

STEP 3. Fix  $t \in \mathbb{C}$ . The second condition of Lemma 3.15 is satisfied since by Lemma 13.4 it holds that

$$\lim_{T \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} |S_n(t) - S_{n, T}(t)|^p = 0.$$

Applying Lemma 3.15, we obtain that (13.18) holds in the sense of finite-dimensional distributions. To complete the proof of (13.18), we need to show that the sequence of random functions  $S_n(t)$  is tight on  $\mathcal{H}(\mathbb{C})$ . The tightness follows from Proposition 13.1 and Proposition 3.12.

**13.4. Proofs of Theorems 2.32, 2.35, 2.37, 2.38.** These proofs follow the method of adjoining the glassy phase levels developed in Sections 10, 11, 12, 13 and do not require any new ideas. For this reason, we will just give the idea of the proofs.

*Idea of proof of Theorems 2.32 and 2.35.* The normalizing sequence  $f_n(\beta_*; t)$  in these theorems is given by

$$(13.20) \quad f_n(\beta_*; t) = \left( \beta_* + \frac{t}{n} \right) \sum_{j=1}^{d_1} \sqrt{na_j} u_{n,j} + \\ + \sum_{j=d_1+1}^{d_1+d_2} \left( \frac{1}{2} \log N_{n,j} + a_j \sigma_*^2 n \right) + \sum_{j=d_1+d_2+1}^d \left( \log N_{n,j} + \frac{1}{2} a_j \beta_*^2 n \right).$$

The proof is by induction over  $d_1$ , the number of glassy phase levels. In the case  $d_1 = 0$ , Theorems 2.32 and 2.35 were already established in Theorems 7.3 and 7.4. Note that in the case  $d_1 = 0$  the first sum in the definition of  $f_n(\beta_*; t)$  vanishes, whereas the remaining two sums are equal to the normalizing constants used in Theorems 7.3 and 7.4, up to a factor of the form  $e^{ic_n + o(1)}$ , where  $c_n$  is real constant. The phase factor  $e^{ic_n}$  can be ignored since the limiting process is isotropic. So, Theorems 7.3 and 7.4 state that in the case  $d_1 = 0$  we have that weakly on  $\mathcal{H}(\mathbb{C})$ ,

$$(13.21) \quad \left\{ e^{-f_n(\beta_*; t)} \mathcal{Z}_n \left( \beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \xrightarrow[n \rightarrow \infty]{w} \{ e^{\lambda' t} N' + e^{\lambda'' t} N'' : t \in \mathbb{C} \},$$

where in the case  $d_2 = 0$  (Theorem 7.4) we have to replace  $N''$  by 1. The proof of Theorems 2.32 and 2.35 in the case  $d_1 \geq 1$  proceeds by adjoining  $d_1$  glassy phase levels to (13.21) one by one as in Sections 11, 12, 13.  $\square$

*Idea of proof of Theorem 2.37.* Let  $\beta \in \mathbb{C}$  be such that  $\sigma = \frac{\sigma_l}{2}$  and  $\tau > \frac{\sigma_l}{2}$  for some  $1 \leq l \leq d$ . The normalizing sequence  $r_n(\beta)$  from Theorem 2.37 is given as follows.

If  $\frac{\sigma_k}{\sqrt{2}} < |\beta| < \frac{\sigma_{k+1}}{\sqrt{2}}$  for some  $l < k \leq d$ , then

$$(13.22) \quad r_n(\beta) = \beta \sum_{j=1}^{l-1} \sqrt{na_j} u_{n,j} + \sum_{j=l}^k \left( \frac{1}{2} \log N_{n,j} + a_j \sigma^2 n \right) + \sum_{j=k+1}^d \left( \log N_{n,j} + \frac{1}{2} \beta^2 n a_j \right).$$

The proof of Theorem 2.37 is by induction over  $l$ . For  $l = 1$  (no glassy phase levels), we proved in Theorem 2.10 that

$$(13.23) \quad \frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \frac{N}{\sqrt{2}},$$

where  $N \sim N_{\mathbb{C}}(0, 1)$ . By Proposition 2.7 (note that  $|\beta| > \frac{\sigma_1}{\sqrt{2}}$  by our assumptions), we can drop  $\mathbb{E}\mathcal{Z}_n(\beta)$  in (13.23). The asymptotic expression for  $\text{Var } \mathcal{Z}_n(\beta)$  given in Proposition 2.6 has the form  $e^{2r_n(\beta) + ic_n + o(1)}$ , where  $c_n \in \mathbb{R}$ . Using the rotational invariance of the complex normal distribution, we obtain that for  $l = 1$ ,

$$\frac{\mathcal{Z}_n(\beta)}{e^{r_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \frac{N}{\sqrt{2}}.$$

This verifies the basis of induction. The rest of the proof consists of adjoining the glassy phase levels by the same method as developed in Section 13.

Note finally that if  $|\beta| = \frac{\sigma_k}{\sqrt{2}}$  for some  $l < k \leq d$ , then we have to add  $\frac{1}{2} \log 2$  to the expression for  $r_n(\beta)$ . This is related to the additional factor of 2 in the asymptotic expression for  $\text{Var } \mathcal{Z}_n(\beta)$  in Proposition 2.6.  $\square$

*Idea of proof of Theorem 2.38.* Let  $\beta \in \mathbb{C}$  be such that  $\sigma = \tau = \frac{\sigma_l}{2}$  for some  $1 \leq l \leq d$ . The normalizing constant  $r_n(\beta)$  in Theorem 2.38 has the same form as in (13.22), with  $k = l$ . The proof is by induction over  $l$ . For  $l = 1$  (no glassy phase levels), we proved in Theorem 2.10 that (13.23) holds. However, this time we cannot drop  $\mathbb{E}\mathcal{Z}_n(\beta)$  since by Propositions 2.5 and 2.6 we have

$$\lim_{n \rightarrow \infty} e^{-i\sigma\tau an} \frac{\mathbb{E}\mathcal{Z}_n(\beta)}{\sqrt{\text{Var } \mathcal{Z}_n(\beta)}} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}\mathcal{Z}_n(\beta)}{e^{r_n(\beta)}} = \frac{1}{\sqrt{2}}.$$

It follows that we can write (13.23) as follows:

$$\frac{\mathcal{Z}_n(\beta)}{e^{r_n(\beta)}} \xrightarrow[n \rightarrow \infty]{d} \frac{N + 1}{\sqrt{2}},$$

where  $N \sim N_{\mathbb{C}}(0, 1)$ . This verifies the basis of induction. The rest of the proof consists of adjoining the glassy phase levels by the same method as developed in Section 13 and Section 12.  $\square$

## 14. LIMITING LOG-PARTITION FUNCTION AND GLOBAL DISTRIBUTION OF ZEROS

**14.1. Limiting log-partition function: Proof of Theorem 2.1.** The idea is that we have already proved a distributional limit theorem for  $\mathcal{Z}_n(\beta)$ , for every  $\beta \in \mathbb{C}$ . From that, we can deduce that  $F_n(\beta) := \frac{1}{n} \log |\mathcal{Z}_n(\beta)|$  converges in probability. The  $L^q$ -convergence will be established later, in Proposition 14.6. The next lemma is taken from [25]; see Lemma 3.9 there.

**Lemma 14.1.** *Let  $Z, Z_1, Z_2, \dots$  be random variables with values in  $\mathbb{C}$  and let  $m_n \in \mathbb{C}$ ,  $v_n \in \mathbb{C} \setminus \{0\}$  be sequences of normalizing constants such that*

$$(14.1) \quad \frac{Z_n - m_n}{v_n} \xrightarrow[n \rightarrow \infty]{d} Z.$$

*The following two statements hold:*

- (1) *If  $|v_n| = o(|m_n|)$  and  $|m_n| \rightarrow \infty$  as  $n \rightarrow \infty$ , then  $\frac{\log |Z_n|}{\log |m_n|} \xrightarrow[n \rightarrow \infty]{P} 1$ .*
- (2) *If  $|m_n| = O(|v_n|)$  and  $|v_n| \rightarrow \infty$  as  $n \rightarrow \infty$  and  $Z$  has no atoms, then  $\frac{\log |Z_n|}{\log |v_n|} \xrightarrow[n \rightarrow \infty]{P} 1$ .*

Note that we can view  $m_n$  as the asymptotic “location” and  $v_n$  as the asymptotic “fluctuations” of  $Z_n$ . There are two parts in the lemma depending on what parameter, location (Part 1), or fluctuations (Part 2), dominates.

*Proof of Theorem 2.1.* Recall that  $p_k(\beta)$ , where  $1 \leq k \leq d$ , is given by (2.6). Let  $p(\beta) = p_1(\beta) + \dots + p_d(\beta)$ . It is easy to check that  $p(\beta) > 0$  for all  $\beta \in \mathbb{C}$ . Our aim is to prove that

$$(14.2) \quad P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = p(\beta).$$

We are going to verify the conditions of Lemma 14.1 for  $Z_n = \mathcal{Z}_n(\beta)$  and suitable  $m_n$ ,  $v_n$ ,  $Z$ . Theorem 2.1 is known for  $\beta \in \mathbb{R}$ , see [11, 16, 7, 9], so in the sequel we always assume that  $\beta \in \mathbb{C} \setminus \mathbb{R}$ . By symmetry, see (1.10), (1.11), we may assume that  $\sigma \geq 0$  and  $\tau > 0$ . We consider three cases.

**CASE 1: Location dominates.** Let  $\beta \in E^d = E_1$ . By Corollary 10.3, we can find a sufficiently small  $\varepsilon = \varepsilon(\beta) > 0$  such that

$$\frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{-\varepsilon n} \mathbb{E}\mathcal{Z}_n(\beta)} \xrightarrow[n \rightarrow \infty]{d} 0.$$

Hence, we can apply Part 1 of Lemma 14.1 with  $m_n = \mathbb{E}\mathcal{Z}_n(\beta)$  to obtain that

$$(14.3) \quad P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathbb{E}\mathcal{Z}_n(\beta)| = \log \alpha + \frac{1}{2}(\sigma^2 - \tau^2)a = p(\beta),$$

where we used Proposition 2.5 and (2.6).

**CASE 2: Fluctuations dominate.** Let  $\beta \in G^{d_1} F^{d_2} E^{d_3}$ , where  $d_3 \neq d$ . By Theorem 2.17, we can apply Part 2 of Lemma 14.1 with  $m_n = 0$  and  $v_n = e^{c_n(\beta)}$ , where  $c_n(\beta)$  is given by (2.24), (2.25). The fact that the limiting variable  $Z$  (which may be of three different types, see Proposition 2.17) has no atoms has been verified in the case  $d_1 > 0$ ,  $d_2 = 0$  in Proposition 8.5 and is trivial in the remaining two cases. It follows that

$$P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Re} c_n(\beta) = p(\beta),$$

where the last step follows by comparing (2.24) with (2.6).

**CASE 3: The boundary case.** Here we assume that  $\beta$  is located on the boundary of some phase  $G^{d_1} F^{d_2} E^{d_3}$ . There are 4 subcases.

**CASE 3A: Beak shaped boundaries.** Assume that  $\beta \in \mathbb{C}$  is such that  $\sigma + \tau = \sigma_l$ ,  $\sigma > \frac{\sigma_l}{2}$ ,  $\tau > 0$ , for some  $1 \leq l \leq d$ .

If  $2 \leq l \leq d$ , then by Theorem 12.11 we can apply Part 2 of Lemma 14.1 with  $m_n = 0$  and  $v_n = e^{\hat{h}_{n,l}(\beta)}$ , where  $\hat{h}_{n,l}(\beta)$  is given by (12.32). Note that the limiting variable  $Z$  is given by a Poisson cascade zeta function with  $l - 1$  variables and has no atoms by Proposition 8.5 (at this point we use that  $l \neq 1$ ). It follows that

$$P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Re} \hat{h}_{n,l}(\beta) = p(\beta),$$

where the last step follows by comparing (12.32) and (2.6).

If  $l = 1$ , then the above argument breaks down since the limiting variable  $Z = 1$  has atoms. However, using (12.33) and (12.34) we see that we can apply Part 1 of Lemma 14.1 with  $m_n = \mathbb{E} \mathcal{Z}_n(\beta)$ . This yields (14.2) by the same computation as in (14.3).

CASE 3B: *Arc shaped boundaries.* Assume that  $\beta \in \mathbb{C}$  is such that for some  $d_1, d_2, d_3 \in \{0, \dots, d\}$  with  $d_1 + d_2 + d_3 = d$  we have

$$\frac{\sigma_{d_1}}{2} < \sigma < \frac{\sigma_{d_1+1}}{2}, \quad \tau > 0, \quad \sigma^2 + \tau^2 = \frac{\sigma_{d_1+d_2}^2}{2}.$$

Due to Theorems 2.32, 2.35, we can apply Part 2 of Lemma 14.1 with  $m_n = 0$  and  $v_n = e^{f_n(\beta; 0)}$ . The limiting random variable  $Z$  has the form

$$Z = \begin{cases} \sqrt{W}N + \zeta^{(d_1)}, & \text{if } d_2 = 1, \\ \sqrt{2W}N, & \text{if } 2 \leq d_2 \leq d; \end{cases}$$

see Theorems 2.32, 2.35. Note that the random variable  $W = \zeta_P(2T^{d_1}(\sigma))$  has no atom at 0 by Proposition 8.5. By Lemma 8.8, the random variable  $Z$  has no atoms. By Part 2 of Lemma 14.1,

$$P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Re} f_n(\beta; 0) = p(\beta),$$

where we used (13.20) and (2.6).

CASE 3C: *Vertical boundaries.* Assume that  $\sigma = \frac{\sigma_l}{2}$  and  $\tau > \frac{\sigma_l}{2}$  for some  $1 \leq l \leq d$ . By Theorem 2.37, we can apply Part 2 of Lemma 14.1 with  $m_n = 0$ ,  $v_n = e^{r_n(\beta)}$ . The limiting random variable  $Z$  has no atoms by Lemma 8.8 and Proposition 8.5. By Part 2 of Lemma 14.1,

$$P\text{-}\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \lim_{n \rightarrow \infty} \frac{1}{n} \operatorname{Re} r_n(\beta) = p(\beta),$$

where we used (13.22) and (2.6).

CASE 3D: *Triple points.* Assume that  $\sigma = \tau = \frac{\sigma_l}{2}$ , for some  $1 \leq l \leq d$ . By Theorem 2.38, we can apply Part 2 of Lemma 14.1 with  $m_n = 0$  and  $v_n = e^{r_n(\beta)}$ . The limiting random variable  $Z$  has no atoms by the same argument as in Case 3B. As in Case 3C it follows that (14.2) holds.  $\square$

**14.2. Estimates for the concentration function and  $L^q$ -convergence.** We will need to bound the probability of the event  $|\mathcal{Z}_n(\beta)| \leq r$ , where  $r \geq 0$  is small. For this purpose, the notion of the concentration function is useful; see, e.g., [35, §1.5]. Denote by  $B_r(t) = \{z \in \mathbb{C} : |z - t| \leq r\}$  the disk of radius  $r \geq 0$  centered

at  $t \in \mathbb{C}$ . Given a random variable  $X$  with values in  $\mathbb{C}$  define its concentration function by

$$(14.4) \quad Q(X; r) = \sup_{t \in \mathbb{C}} \mathbb{P}[X \in B_r(t)], \quad r \geq 0.$$

The next fact follows immediately from the convolution formula and can be found in [35, §1.5, Lemma 1.11]: If  $Y_1, \dots, Y_m$  are independent random variables with values in  $\mathbb{C}$ , then

$$(14.5) \quad Q(Y_1 + \dots + Y_m; r) \leq \min_{i=1, \dots, m} Q(Y_i; r).$$

**Lemma 14.2.** *Let  $X$  and  $Y$  be independent random values with values in  $\mathbb{C}$ . Then, for every  $r \geq 0$ ,*

$$Q(XY; r) \leq Q(X; \sqrt{r}) + \mathbb{P}[|Y| \leq \sqrt{r}].$$

*Proof.* Let  $t \in \mathbb{C}$ . Then,

$$\mathbb{P}[XY \in B_r(t)] \leq \mathbb{P}[XY \in B_r(t), |Y| > \sqrt{r}] + \mathbb{P}[|Y| \leq \sqrt{r}].$$

Let  $\mu_Y$  be the distribution of the random variable  $Y$ . Conditioning on  $Y = w$ , where  $w \in \mathbb{C}$ , and using the formula for the total probability, we get

$$\mathbb{P}[XY \in B_r(t), |Y| > \sqrt{r}] = \int_{\mathbb{C} \setminus B_{\sqrt{r}}(0)} \mathbb{P}[wX \in B_r(t)] \mu_Y(dw) \leq Q(X; \sqrt{r}),$$

where the last inequality holds since we have  $\mathbb{P}[wX \in B_r(t)] \leq Q(X; \sqrt{r})$  as long as  $|w| > \sqrt{r}$ . The required inequality follows.  $\square$

**Lemma 14.3.** *Let  $K \subset \mathbb{C} \setminus \{0\}$  be a compact set and let  $\varepsilon > 0$ . Let  $\xi$  be a real standard normal random variable. Then, there exist constants  $C > 0$ ,  $N \in \mathbb{N}$ ,  $\delta > 0$  (which depend on  $K$  and  $\varepsilon$ ) such that for every  $\beta \in K$ ,  $n > N$ ,  $r \in (0, e^{-\varepsilon n})$  we have*

$$Q(e^{\beta \sqrt{n} \xi}; r) < Cr^\delta.$$

*Proof.* See Eq. (3.35) in [25].  $\square$

**Lemma 14.4.** *Let  $K \subset \mathbb{C} \setminus \{0\}$  be a compact set and let  $\varepsilon > 0$ . Then, there exist constants  $C > 0$ ,  $N \in \mathbb{N}$ ,  $\delta > 0$  (which depend on  $K$  and  $\varepsilon$ ) such that, for every  $\beta \in K$ ,  $n > N$ ,  $r \in (0, e^{-\varepsilon n})$ , we have*

$$Q(\mathcal{Z}_n(\beta); r) < Cr^\delta.$$

*Proof.* We will prove this by induction over  $d$ , the number of GREM levels. If  $d > 1$ , then we assume that the statement of the lemma is true for the GREM with  $d - 1$  levels. For  $d = 1$ , we don't need any assumption. For the partition function of the GREM with  $d$  levels, we have a representation

$$\mathcal{Z}_n(\beta) = \sum_{k=1}^{N_{n,1}} e^{\beta \sqrt{n a_1} \xi_k} \mathcal{Z}_{n,k}^*(\beta).$$

Here, for every  $k = 1, \dots, N_{n,1}$ ,  $\mathcal{Z}_{n,k}^*(\beta)$  is an analogue of  $\mathcal{Z}_n(\beta)$  with  $d - 1$  levels instead of  $d$  levels. Assume first that  $d > 1$ . By Lemma 14.2,

$$Q(e^{\beta \sqrt{n a_1} \xi_1} \mathcal{Z}_{n,1}^*(\beta); r) \leq Q(e^{\beta \sqrt{n a_1} \xi_1}; \sqrt{r}) + \mathbb{P}[|\mathcal{Z}_{n,1}^*(\beta)| \leq \sqrt{r}].$$

The first term is bounded by  $C_1 r^{\delta_1}$  by Lemma 14.3 (in which we take  $\varepsilon/2$  instead of  $\varepsilon$ ). The second term is bounded by  $C_2 r^{\delta_2}$  by the induction assumption. Hence, for every sufficiently large  $n$  and all  $\beta \in K$ ,  $r \in (0, e^{-\varepsilon n})$ , we have

$$Q(e^{\beta\sqrt{na_1}\xi_1} \mathcal{Z}_{n,1}^*(\beta); r) \leq C_3 r^{\delta_3}.$$

This inequality is true in the case  $d = 1$ , too, because in this case  $\mathcal{Z}_n^*(\beta) = 1$  and we can directly use Lemma 14.3. For independent random variables  $Y_1, \dots, Y_m$ , we have  $Q(Y_1 + \dots + Y_m; r) \leq Q(Y_1; r)$ ; see (14.5). Hence, we obtain that

$$Q(\mathcal{Z}_n(\beta); r) \leq Q(e^{\beta\sqrt{na_1}\xi_1} \mathcal{Z}_{n,1}^*(\beta); r) \leq C_3 r^{\delta_3}.$$

This completes the induction.  $\square$

Let  $F_n(\beta) = \frac{1}{n} \log |\mathcal{Z}_n(\beta)|$  and recall that  $p(\beta) = \sum_{k=1}^d p_k(\beta)$ , where  $p_k(\beta)$  has been defined in (2.6).

**Lemma 14.5.** *Let  $K \subset \mathbb{C} \setminus \{0\}$  be a compact set and let  $r > 0$ . Then, we can find  $C > 0$  and  $N \in \mathbb{N}$  depending on  $K$  and  $r$  such that for all  $n > N$ ,*

$$\sup_{\beta \in K} \mathbb{E} |F_n(\beta)|^r < C.$$

*Proof.* For  $u > 0$  and  $\beta \in K$  we have

$$\mathbb{P}[F_n(\beta) > u] = \mathbb{P}[|\mathcal{Z}_n(\beta)| > e^{nu}] \leq e^{-nu} \mathbb{E} |\mathcal{Z}_n(\beta)| \leq e^{-nu} N_n \mathbb{E} e^{\sigma\sqrt{na}\xi} \leq e^{(C-u)n},$$

where  $C = C(K)$ . Consequently, for all  $\beta \in K$ ,  $u > C$ ,  $n \in \mathbb{N}$  we have

$$\mathbb{P}[F_n(\beta) > u] \leq e^{C-u}.$$

To complete the proof, we need to estimate the lower tail of  $F_n(\beta)$ . By Lemma 14.4, we can find  $C > 0$ ,  $\delta > 0$ ,  $N \in \mathbb{N}$  such that for all  $\beta \in K$ ,  $u > 1$  and  $n > N$ ,

$$\mathbb{P}[F_n(\beta) < -u] = \mathbb{P}[|\mathcal{Z}_n(\beta)| < e^{-un}] < Q(\mathcal{Z}_n(\beta); e^{-un}) < C e^{-\delta un} < C e^{-\delta u}.$$

The last two displays imply the claim.  $\square$

We have already shown in Section 14.1 that for every  $\beta \in \mathbb{C}$ ,  $F_n(\beta)$  converges to  $p(\beta)$  in probability. Now we are able to prove the  $L^q$ -convergence.

**Proposition 14.6.** *Fix  $q \geq 1$ . For every  $\beta \in \mathbb{C}$ ,  $F_n(\beta)$  converges to  $p(\beta)$  in  $L^q$ .*

*Proof.* The statement is trivial for  $\beta = 0$ , so fix some  $\beta \neq 0$ . We need to show that the random variables  $|F_n(\beta)|^q$ ,  $n \in \mathbb{N}$ , are uniformly integrable; see [26, Proposition 3.12]. For this, it suffices to show that the sequence  $F_n(\beta)$  is bounded in  $L^r$ , for some  $r > q$ ; see [26, p. 44]. This fact has already been established in Lemma 14.5.  $\square$

**14.3. Global distribution of zeros: Proof of Theorem 2.3.** The proof is analogous to the proof of Theorem 2.1 in [25].

**STEP 1.** We need to show that for every infinitely differentiable, compactly supported function  $f: \mathbb{C} \rightarrow \mathbb{R}$ ,

$$(14.6) \quad \frac{1}{n} \sum_{\beta \in \mathbb{C}: \mathcal{Z}_n(\beta)=0} f(\beta) \xrightarrow[n \rightarrow \infty]{P} \frac{1}{2\pi} \int_{\mathbb{C}} f(\beta) d\Xi(\beta).$$

This is equivalent to the statement of Theorem 2.3 by [26, Theorem 14.16]. When proving (14.6) we can assume that  $f$  vanishes in some neighborhood of the origin. To see this, note that we can write  $f = f_1 + f_2$ , where  $f_1, f_2: \mathbb{C} \rightarrow \mathbb{R}$  are compactly



supported and infinitely differentiable,  $f_1$  vanishes in the disk  $\{|z| \leq \frac{\sigma_1}{4}\}$ , while  $f_2$  vanishes outside the disk  $\{|z| \leq \frac{\sigma_1}{3}\}$ . Since  $f_2$  does not contribute neither to the left-hand side of (14.6) (with probability approaching 1, by Theorem 2.23), nor to the right-hand side of (14.6) (by the definition of  $\Xi$ ), we can and will assume that  $f = f_1$ .

STEP 2. Let  $\lambda$  be the Lebesgue measure on  $\mathbb{C}$ . By the Poincaré–Lelong formula, see [20, §2.4.1],

$$(14.7) \quad \frac{1}{n} \sum_{\beta \in \mathbb{C}: \mathcal{Z}_n(\beta)=0} f(\beta) = \frac{1}{2\pi n} \int_{\mathbb{C}} \log |\mathcal{Z}_n(\beta)| \Delta f(\beta) \lambda(d\beta).$$

Recall that by Theorem 2.1 the random variable  $F_n(\beta) := \frac{1}{n} \log |\mathcal{Z}_n(\beta)|$  converges to  $p(\beta) = \sum_{k=1}^d p_k(\beta)$  in  $L^1$  for every  $\beta \in \mathbb{C}$ . From (14.9) (which will be established in Step 3 below), we conclude that Theorem 2.3 is equivalent to

$$\int_{\mathbb{C}} F_n(\beta) \Delta f(\beta) \lambda(d\beta) \xrightarrow[n \rightarrow \infty]{P} \int_{\mathbb{C}} p(\beta) \Delta f(\beta) \lambda(d\beta).$$

We will show that this holds even in  $L^1$ . By Fubini's theorem, it suffices to show that

$$(14.8) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{C}} \mathbb{E} |F_n(\beta) - p(\beta)| |\Delta f(\beta)| \lambda(d\beta) = 0.$$

We know from Theorem 2.1 that  $\lim_{n \rightarrow \infty} \mathbb{E} |F_n(\beta) - p(\beta)| = 0$ , for every  $\beta \in \mathbb{C}$ . To complete the proof, we need to interchange the limit and the integral. Recalling that  $f$  vanishes on a neighborhood of 0 and applying Lemma 14.5 we obtain that there is  $C > 0$  such that for all  $\beta \in \text{supp } f$  and all sufficiently large  $n \in \mathbb{N}$ ,

$$\mathbb{E} |F_n(\beta) - p(\beta)| |\Delta f(\beta)| < C.$$

This justifies the use of the dominated convergence theorem and completes the proof of (14.8).

STEP 3. In this step, we show that  $\Delta p = \Xi$  in the sense of generalized functions. This means that for every compactly supported infinitely differentiable function  $f: \mathbb{C} \rightarrow \mathbb{R}$ ,

$$(14.9) \quad \int_{\mathbb{C}} p(\beta) \Delta f(\beta) \lambda(d\beta) = \int_{\mathbb{C}} f(\beta) \Xi(d\beta).$$

Here,  $p(\beta) = \sum_{k=1}^d p_k(\beta)$ , where  $p_k(\beta)$  is given by (2.6), and  $\Xi = \sum_{k=1}^d \Xi_k$ , where  $\Xi_k = \Xi_k^F + \Xi_k^{EF} + \Xi_k^{EG}$  and the three terms were described in Section 2.3.

It suffices to show that  $\Delta p_k = \Xi_k$  for every  $1 \leq k \leq d$ . This means that

$$(14.10) \quad \int_{\mathbb{C}} p_k(\beta) \Delta f(\beta) \lambda(d\beta) = \int_{\mathbb{C}} f(\beta) \Xi_k(d\beta).$$

This computation has been performed by Derrida [15] (who has  $a_k = \frac{1}{2}$ ,  $\alpha_k = \log 2$ ), but for completeness we provide the details. Green's second identity applied to the domains  $B = E_k, F_k, G_k$  gives

$$\begin{aligned} & \int_B p_k(\beta) \Delta f(\beta) \lambda(d\beta) \\ &= \int_B \Delta p_k(\beta) f(\beta) \lambda(d\beta) + \oint_{\partial B} \left( f(\beta) \frac{\partial p_k(\beta)}{\partial \mathbf{n}} - p_k(\beta) \frac{\partial f(\beta)}{\partial \mathbf{n}} \right) |d\beta|. \end{aligned}$$

Here,  $\mathbf{n}$  denotes the unit inward pointing normal to the boundary of  $B$  and  $\frac{\partial}{\partial \mathbf{n}}$  is the corresponding directional derivative. Adding these three identities, noting that the pointwise Laplacian of  $p_k$  is given by

$$\Delta p_k(\beta) = \begin{cases} 2a_k, & \text{if } \beta \in F_k, \\ 0, & \text{if } \beta \in E_k \cup G_k, \end{cases}$$

and that the terms involving  $\frac{\partial f(\beta)}{\partial \mathbf{n}}$  cancel (by the continuity of  $p_k$ ), we obtain

$$\int_{\mathbb{C}} p_k(\beta) \Delta f(\beta) \lambda(d\beta) = 2a_k \int_{F_k} f(\beta) \lambda(d\beta) + \oint_{\gamma} f(\beta) \left( \frac{\partial}{\partial \mathbf{n}_+} + \frac{\partial}{\partial \mathbf{n}_-} \right) p_k(\beta) |d\beta|.$$

Here,  $\gamma$  is the union of lines and arcs which constitute the boundaries of  $E_k, F_k, G_k$  and  $\mathbf{n}_+$  and  $\mathbf{n}_-$  denote the unit normals to  $\gamma$  (with directions opposite to each other). To complete the proof, we need to compute the jump of the normal derivative of  $p_k$ :

$$(14.11) \quad \left( \frac{\partial}{\partial \mathbf{n}_+} + \frac{\partial}{\partial \mathbf{n}_-} \right) p_k(\beta).$$

There are three cases.

CASE EF. On the boundary of  $E_k$  and  $F_k$  (two circular arcs), (14.11) equals

$$\frac{\sqrt{2}}{\sigma_k} \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right) \left( \frac{1}{2} \log \alpha_k + a_k \sigma^2 - \log \alpha_k - \frac{1}{2} a_k (\sigma^2 - \tau^2) \right) = \sqrt{a_k \log \alpha_k}.$$

CASE EG. On the boundary of  $E_k$  and  $G_k$  (four line segments), (14.11) equals

$$\frac{1}{\sqrt{2}} \left( \operatorname{sgn} \sigma \frac{\partial}{\partial \sigma} + \operatorname{sgn} \tau \frac{\partial}{\partial \tau} \right) \left( |\sigma| \sqrt{2a_k \log \alpha_k} - \log \alpha_k - \frac{1}{2} a_k (\sigma^2 - \tau^2) \right) = \sqrt{2a_k} |\tau|.$$

CASE FG. On the boundary of  $F_k$  and  $G_k$  (four half-lines), (14.11) equals

$$\left( \operatorname{sgn} \sigma \frac{\partial}{\partial \sigma} \right) \left( |\sigma| \sqrt{2a_k \log \alpha_k} - \frac{1}{2} \log \alpha_k - a_k \sigma^2 \right) = 0.$$

Combining everything together, we obtain that  $\int_{\mathbb{C}} p_k(\beta) \Delta f(\beta) \lambda(d\beta)$  is given by

$$2a_k \int_{F_k} f(\beta) \lambda(d\beta) + \sqrt{a_k \log \alpha_k} \oint_{\bar{E}_k \cap \bar{F}_k} f(\beta) |d\beta| + \sqrt{2a_k} \oint_{\bar{E}_k \cap \bar{G}_k} |\tau| f(\beta) |d\beta|.$$

This coincides with  $\int_{\mathbb{C}} f(\beta) \Xi_k(d\beta)$  by definition of  $\Xi_k$ , see Section 2.3. The proof of (14.10) is complete.

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